Laplace Transform Summary

Definition 7.1.1
$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = F(s)$

Theorem 7.1.1

a) $\mathcal{L}\{1\} = \frac{1}{s}$
b) $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$
c) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$
d) $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$
e) $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$
f) $\mathcal{L}\{\sinh kt\} = \frac{k}{s^2-k^2}$
g) $\mathcal{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$

Theorem 7.2.1

a) $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
b) $\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$
c) $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
d) $\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$
e) $\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$
f) $\mathcal{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} = \sinh kt$
g) $\mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh kt$

Theorem 7.2.2 (particular cases)
$\mathcal{L}\{y'\} = sY(s) - y(0)$, $\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$

Theorem 7.3.1 (First Translation Theorem)
$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

Definition 7.3.1 (Unit step function or Heaviside function)
$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

Theorem 7.3.2 (Second Translation Theorem)
$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$ OR $\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$
Inverse form: $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)$
$\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s}$

Theorem 7.4.1
$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$

Definition: (Convolution)
$f * g = \int_0^t f(\tau)g(t-\tau)d\tau$

Theorem 7.4.2
$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$
Inverse form: $\mathcal{L}^{-1}\{F(s)G(s)\} = f * g$

Transform of an integral

$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$

Inverse form: $\int_0^t f(\tau)d\tau = \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\}$

Theorem 7.4.3

$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-st}}\int_0^T e^{-st}f(t)dt$, where f is periodic with period T (that is, $f(t+T) = f(t)$)

Definition: (Unit impulse function)

$$\delta_a(t-t_0) = \begin{cases} 0, & 0 \le t < t_0 - a \\ \dfrac{1}{2a}, & t_0 \le t < t_0 + a \\ 0, & t \ge t_0 + a \end{cases}$$

Definition: (Dirac delta function)

$\delta(t-t_0) = \lim_{a\to 0}\delta_a(t-t_0)$

Theorem 7.5.1 (Transform of the Dirac delta function)

For $t_0 > 0$, $\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$