8.4 - Matrix Exponential

8.4a

Recall that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$ This is the power series representation for the exponential function. In the same way, we can define a matrix exponential.

Definition: For any $n \times n$ matrix $\mathbf{A}$,

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \cdots + \mathbf{A}^k\frac{t^k}{k!} + \cdots = \sum_{k=0}^{\infty} \mathbf{A}^k\frac{t^k}{k!}$$

Example: Compute $e^{\mathbf{A}t}$ and $e^{-\mathbf{A}t}$.

$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$A^2 = AA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$A^3 = A^2A = IA = A, \quad A^4 = A^2A^2 = II = I$

$e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}t^2 & 0 \\ 0 & \frac{1}{2}t^2 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{3!}t^3 \\ \frac{1}{3!}t^3 & 0 \end{pmatrix} + \cdots$

$e^{At} = \begin{pmatrix} 1 + \frac{1}{2}t^2 + \frac{1}{4!}t^4 + \cdots & t + \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \cdots \\ t + \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \cdots & 1 + \frac{1}{2}t^2 + \frac{1}{4!}t^4 + \cdots \end{pmatrix}$

$e^{At} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$

In general, $AB \neq BA$. But if it happens that $AB = BA$, then $e^{A+B} = e^{AB}$.

So $e^{At}e^{-At} = e^{At-At} = e^{0} = I$ ← matrix

$\Rightarrow e^{-At} = (e^{At})^{-1}$

Here, for e^{-At}, we get $e^{-At} = \begin{pmatrix} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{pmatrix}$

8.4b

$\vec{x} = e^{At}\vec{c}$

Consider $\mathbf{X}' = \mathbf{AX}$. If $\mathbf{X} = e^{\mathrm{A}t}\mathrm{C}$, then $\mathbf{X}' = \frac{d}{dt}\left(e^{\mathrm{A}t}\mathrm{C}\right)$. To find $\frac{d}{dt}\left(e^{\mathrm{A}t}\right)$, we use the definition:

$$\frac{d}{dt}\left(e^{\mathrm{A}t}\right) = \frac{d}{dt}\left(\mathrm{I} + \mathrm{A}t + \mathrm{A}^2\frac{t^2}{2!} + \mathrm{A}^3\frac{t^3}{3!} + \cdots\right)$$

$= A + A^2 t + A^3 \frac{t^2}{2!} + \cdots$

$= A\left(I + At + A^2\frac{t^2}{2!} + \cdots\right)$

$\frac{d}{dt}\left(e^{At}\right) = A\,e^{At}$

So if $\vec{X} = e^{At}\vec{C}$, then

$\mathbf{X}' = \mathbf{A}e^{\mathbf{A}t}\mathbf{C}$

So for $\mathbf{X}' = \mathbf{AX}$, with $\mathbf{A}$ containing constant entries, $\mathbf{X} = e^{\mathbf{A}t}\mathbf{C}$ is a solution. (Note: $\mathbf{C}$ is a column matrix of arbitrary coefficients.)

Example: Use the matrix exponential to find the general solution of the given system.

$$\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{X}$$

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ from 1st example.

We found $e^{At} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$

So $\vec{X}(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\vec{C}$

Initial-value problems / nonhomogeneous systems

$e^{\mathbf{A}t} = \mathbf{\Phi}$ is the fundamental matrix for the system, so variation of parameters yields, the general solution

$$\mathbf{X} = e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{F}(s)ds$$

Note: $e^{-\mathbf{A}s}$ is $e^{\mathbf{A}t}$ with t replaced by $-s$. In our work, we will take $t_0 = 0$.

Example: Find the general solution of the given system.

$$\mathbf{X}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

$$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix}$$

Prediction: $A^4 = \begin{pmatrix} 1^4 & 0 \\ 0 & 2^4 \end{pmatrix}$

$$e^{At} = I + \begin{pmatrix} t & 0 \\ 0 & 2t \end{pmatrix} + \frac{1}{2}\begin{pmatrix} t^2 & 0 \\ 0 & 4t^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} t^3 & 0 \\ 0 & 8t^3 \end{pmatrix} + \cdots$$

$$e^{At} = \begin{pmatrix} 1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3 + \cdots & 0 \\ 0 & 1 + (2t) + \frac{1}{2}(2t)^2 + \frac{1}{3!}(2t)^3 + \cdots \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$$

$$\Phi(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \Rightarrow \vec{X}_c = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}\vec{C}$$

$$e^{-As} = \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{-2s} \end{pmatrix}$$

$$\int_0^t \Phi^{-1}(s)\vec{F}(s)\,ds = \int_0^t e^{-As}\vec{F}(s)\,ds$$

← our author's convention

$$= \int_0^t \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{-2s} \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} ds$$

$$= \int_0^t \begin{pmatrix} 3e^{-s} \\ -e^{-2s} \end{pmatrix} ds = \begin{pmatrix} -3e^{-s} \\ \frac{1}{2}e^{-2s} \end{pmatrix} \Bigg|_0^t$$

$$= \begin{pmatrix} -3e^{-t} + 3 \\ \frac{1}{2}e^{-2t} - \frac{1}{2} \end{pmatrix}$$

$$e^{At} \int_0^t e^{-As} \vec{F}(s) ds = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} -3e^{-t} + 3 \\ \frac{1}{2}e^{-2t} - \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -3 + 3e^t \\ \frac{1}{2} - \frac{1}{2}e^{2t} \end{pmatrix}$$

$$\vec{x}(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \vec{C} + \begin{pmatrix} -3 + 3e^t \\ \frac{1}{2} - \frac{1}{2}e^{2t} \end{pmatrix}$$

$$\vec{x}(t) = \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix} + \begin{pmatrix} -3 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 3e^t \\ -\frac{1}{2}e^{2t} \end{pmatrix}$$

Let $c_3 = c_1 + 3$ and $c_4 = c_2 - \frac{1}{2}$

$$\boxed{\vec{x}(t) = \begin{pmatrix} c_3 e^t \\ c_4 e^{2t} \end{pmatrix} + \begin{pmatrix} -3 \\ 1/2 \end{pmatrix}}$$