8.3 - Nonhomogeneous Linear Systems

$\mathbf{X}' = \mathbf{AX} + \mathbf{F}(t)$ is a nonhomogeneous system if $\mathbf{F}(t) \neq \mathbf{0}$.

Example: Use the method of undetermined coefficients to solve the given nonhomogeneous system.

$$\mathbf{X}' = \begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 4t + 9e^{6t} \\ -t + e^{6t} \end{pmatrix} \qquad \vec{X}_p = \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} e^{6t}$$

$$\begin{vmatrix} 1-\lambda & -4 \\ 4 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 2\lambda + 17 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 = -16$$

$$(\lambda - 1)^2 = \pm 4i \Rightarrow \lambda = 1 \pm 4i$$

$$\begin{pmatrix} -4i & -4 \\ 4 & -4i \end{pmatrix}\vec{K} = \vec{0} \Rightarrow k_1 = ik_2$$

Let $k_2 = -i \Rightarrow k_1 = 1$

$$\vec{K} = \vec{B}_1 + \vec{B}_2 i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} i$$

$$\vec{X}_c = c_1 e^t \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos 4t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin 4t \right] + c_2 e^t \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos 4t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin 4t \right]$$

Check $\vec{X}_c$ for redundancy w/ $\vec{X}_p$. If there is, stop. Use VoP.

Here, $\vec{F}(t) = \begin{pmatrix} 4t + 9e^{6t} \\ -t + e^{6t} \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} t + \begin{pmatrix} 9 \\ 1 \end{pmatrix} e^{6t}$

linear exponential

$$\vec{X}_p = \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} e^{6t}$$

$$= \begin{pmatrix} a_3 + a_2 t + a_1 e^{6t} \\ b_3 + b_2 t + b_1 e^{6t} \end{pmatrix}$$

$$A\vec{X} = \begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} a_3 + a_2 t + a_1 e^{6t} \\ b_3 + b_2 t + b_1 e^{6t} \end{pmatrix}$$

$$= \begin{pmatrix} a_3 + a_2 t + a_1 e^{6t} - 4b_3 - 4b_2 t - 4b_1 e^{6t} \\ 4a_3 + 4a_2 t + 4a_1 e^{6t} + b_3 + b_2 t + b_1 e^{6t} \end{pmatrix}$$

$$\vec{X}' = A\vec{X} + \vec{F}$$

$$\begin{pmatrix} a_2 + 6a_1 e^{6t} \\ b_2 + 6b_1 e^{6t} \end{pmatrix} = \begin{pmatrix} a_3 + a_2 t + a_1 e^{6t} - 4b_3 - 4b_2 t - 4b_1 e^{6t} + 4t + 9e^{6t} \\ 4a_3 + 4a_2 t + 4a_1 e^{6t} + b_3 + b_2 t + b_1 e^{6t} - t + e^{6t} \end{pmatrix}$$

Constants: $a_2 = a_3 - 4b_3$ $\quad b_2 = 4a_3 + b_3$

t-terms: $0 = a_2 - 4b_2 + 4$ $\quad 0 = 4a_2 + b_2 - 1$

exp: $6a_1 = a_1 - 4b_1 + 9$ $\quad 6b_1 = 4a_1 + b_1 + 1$

. . .

We get

$$\vec{X}(t) = e^t \begin{pmatrix} \cos 4t & \sin 4t \\ \sin 4t & -\cos 4t \end{pmatrix} \vec{C} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t + \begin{pmatrix} 4/17 \\ 1/17 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{6t}$$

Inverses of matrices: Let A be an $n \times n$ matrix.

If $\det(A) \neq 0$, then A^{-1} exists.

Property/def The inverse of A has the property that $AA^{-1} = A^{-1}A = I$.

2×2: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc \neq 0$.

$$\Rightarrow A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

$$A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \Rightarrow \det(A) = 1$$

$$A^{-1} = \frac{1}{1} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}.$$

$$\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

From 8.2

$$\vec{X}(t) = c_1\begin{pmatrix}2\\-1\end{pmatrix}e^{t} + c_2\begin{pmatrix}1\\1\end{pmatrix}e^{4t}$$

$$= \begin{pmatrix}2e^{t}\\-e^{t}\end{pmatrix}c_1 + \begin{pmatrix}e^{4t}\\e^{4t}\end{pmatrix}c_2 = \begin{pmatrix}2c_1e^{t} + c_2e^{4t}\\-c_1e^{t} + c_2e^{4t}\end{pmatrix}$$

$$= \begin{pmatrix}2e^{t} & e^{4t}\\-e^{t} & e^{4t}\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}$$

Phi

Variation of parameters

The complementary solution is $\mathbf{X}_c(t) = \Phi(t)\mathbf{C}$.

Definition: $\Phi(t)$ is the **fundamental matrix** of the system.

$\det(\Phi)$ is W, the Wronskian

For a system with independent solutions, the Wronskian is nonzero, so Φ^{-1} exists.

Since Φ is a solution to $\mathbf{X}' = \mathbf{A}\mathbf{X}$, we have $\Phi' = \mathbf{A}\Phi$. Suppose $\mathbf{X}_p = \Phi(t)\mathbf{U}(t)$, where $\mathbf{U}(t)$ is a column matrix of unknown functions.

$\vec{X}_p' = \Phi'\vec{u} + \Phi\vec{u}'$ into $\vec{X}' = A\vec{X} + \vec{F}$:

$\Phi'\vec{u} + \Phi\vec{u}' = A\Phi\vec{u} + \vec{F}$

$A\Phi\vec{u} + \Phi\vec{u}' = A\Phi\vec{u} + \vec{F}$

$\vec{u}' = \Phi^{-1}\vec{F} \Rightarrow \vec{u} = \int \Phi^{-1}\vec{F}\,dt$

$$\vec{X}_p = \Phi \int \Phi^{-1} \vec{F}\, dt$$

$$\vec{X}_c + \vec{X}_p$$

$$\mathbf{X}(t) = \Phi(t)\mathbf{C} + \Phi(t) \int \Phi^{-1}(t)\mathbf{F}(t)\, dt$$

Example: Use variation of parameters to solve the given nonhomogeneous system.

$$\frac{dx}{dt} = 2x - y$$
$$\frac{dy}{dt} = 3x - 2y + 4t$$

$$= \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \vec{X} + \begin{pmatrix} 0 \\ 4t \end{pmatrix}$$

$\det(A - \lambda I) = 0 \Rightarrow \lambda = 1, -1$

and $\vec{K}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{K}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$$\vec{X} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$$

$$\Phi = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix} \quad \det(\Phi) = 3 - 1 = 2$$

$$\Phi^{-1} = \frac{1}{2} \begin{pmatrix} 3e^{-t} & -e^{-t} \\ -e^t & e^t \end{pmatrix}$$

$$\int \Phi^{-1} \vec{F}\, dt = \frac{1}{2} \int \begin{pmatrix} 3e^{-t} & -e^{-t} \\ -e^t & e^t \end{pmatrix} \begin{pmatrix} 0 \\ 4t \end{pmatrix} dt$$

$$= \frac{1}{2} \int \begin{pmatrix} -4te^{-t} \\ 4te^t \end{pmatrix} dt = \begin{pmatrix} 2te^{-t} + 2e^{-t} \\ 2te^t - 2e^t \end{pmatrix}$$

$$\Phi \int \Phi^{-1}\vec{F}\,dt = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}\begin{pmatrix} 2te^{-t}+2e^{-t} \\ 2te^t-2e^t \end{pmatrix}$$

$$\begin{pmatrix} 2t+2+2t-2 \\ 2t+2+6t-6 \end{pmatrix} = \begin{pmatrix} 4t \\ 8t-4 \end{pmatrix} = \vec{X}_p$$

$$\vec{X}(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}\vec{C} + \begin{pmatrix} 4t \\ 8t-4 \end{pmatrix}$$

Example: Use variation of parameters to solve the given nonhomogeneous system.

$$\mathbf{X}' = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix}\mathbf{X} + \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix} e^{2t}$$

We would find that $\lambda = 2 \pm 2i$ and

$\vec{K} = \vec{B}_1 + \vec{B}_2 i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} i$

This gives us

$\vec{X}(t) = c_1 e^{2t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos 2t - \begin{pmatrix} 0 \\ -2 \end{pmatrix}\sin 2t\right] + c_2 e^{2t}\left[\begin{pmatrix} 0 \\ -2 \end{pmatrix}\cos 2t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin 2t\right]$

$\vec{X}(t) = \Phi\vec{C} = \begin{pmatrix} e^{2t}\cos 2t & e^{2t}\sin 2t \\ 2e^{2t}\sin 2t & -2e^{2t}\cos 2t \end{pmatrix}\vec{C}$

$\Phi^{-1} = \frac{1}{\det(\Phi)}\begin{pmatrix} a & -b \\ -c & a \end{pmatrix}$

$\det(\Phi) = -2e^{4t}\cos^2 2t - 2e^{4t}\sin^2 2t = -2e^{4t}$

$\Phi^{-1} = -\frac{1}{2e^{4t}}\begin{pmatrix} -2e^{2t}\cos 2t & -e^{2t}\sin 2t \\ -2e^{2t}\sin 2t & e^{2t}\cos 2t \end{pmatrix}$

$= \frac{1}{2e^{2t}}\begin{pmatrix} 2\cos 2t & \sin 2t \\ 2\sin 2t & -\cos 2t \end{pmatrix}$

$\Phi^{-1}\vec{F} = \frac{1}{2e^{2t}}\begin{pmatrix} 2\cos 2t & \sin 2t \\ 2\sin 2t & -\cos 2t \end{pmatrix}\begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix} e^{2t}$

$= \frac{1}{2}\begin{pmatrix} 4\cos 2t\sin 2t \\ 2\sin^2 2t - 2\cos^2 2t \end{pmatrix} = \begin{pmatrix} \sin 4t \\ -\cos 4t \end{pmatrix}$

($4\cos 2t\sin 2t = 2\sin 4t$; $2\sin^2 2t - 2\cos^2 2t = -2\cos 4t$)

$$\int \Phi^{-1}\vec{F}\,dt = \int \begin{pmatrix} \sin 4t \\ -\cos 4t \end{pmatrix} dt = \begin{pmatrix} -\frac{1}{4}\cos 4t \\ -\frac{1}{4}\sin 4t \end{pmatrix}$$

$$\Phi \int \Phi^{-1}\vec{F}\,dt = e^{2t}\begin{pmatrix} \cos 2t & \sin 2t \\ 2\sin 2t & -2\cos 2t \end{pmatrix}\begin{pmatrix} -\frac{1}{4}\cos 4t \\ -\frac{1}{4}\sin 4t \end{pmatrix}$$

$$\vec{X}_p = e^{2t}\begin{pmatrix} -\frac{1}{4}(\cos 2t\cos 4t + \sin 2t \sin 4t) \\ -\frac{1}{2}(\sin 2t\cos 4t - \cos 2t \sin 4t) \end{pmatrix}$$

$$\vec{X}(t) = \Phi\vec{C} + \vec{X}_p$$

show-off move

$$\vec{X}(t) = e^{2t}\begin{pmatrix} \cos 2t & \sin 2t \\ 2\sin 2t & -2\cos 2t \end{pmatrix}\vec{C} + e^{2t}\begin{pmatrix} -\frac{1}{4}\cos 2t \\ \frac{1}{2}\sin 2t \end{pmatrix}$$

For an initial-value problem of the form $\mathbf{X}' = \mathbf{AX} + \mathbf{F}(t)$, $\mathbf{X}(t_0) = \mathbf{X}_0$, we have $\mathbf{X} = \mathbf{\Phi C} + \mathbf{\Phi}\int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{F}(s)\,ds \implies \mathbf{X}(t_0) = \mathbf{\Phi}(t_0)\mathbf{C} \implies \mathbf{C} = \mathbf{\Phi}^{-1}(t_0)\mathbf{X}(t_0)$. Thus,

$$\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{X}(t_0) + \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{F}(s)\,ds$$

Example: Solve the given initial-value problem.

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 1/t \\ 1/t \end{pmatrix}, \quad \mathbf{X}(1) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 = 0 \Rightarrow \lambda = 0 \text{ (repeated)}$$

$\lambda = 0$: $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\vec{k} = \vec{0} \Rightarrow k_1 = k_2 \Rightarrow \vec{k} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{p}$: $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\vec{p} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow p_1 - p_2 = 1 \Rightarrow \vec{p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$$

$$\Phi = \begin{pmatrix} 1 & t+1 \\ 1 & t \end{pmatrix} \Rightarrow \det(\Phi) = -1$$

$$\Phi^{-1} = -\begin{pmatrix} t & -t-1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -t & t+1 \\ 1 & -1 \end{pmatrix}$$

$$\Phi^{-1}\vec{F} = \begin{pmatrix} -t & t+1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1/t \\ 1/t \end{pmatrix} = \begin{pmatrix} 1/t \\ 0 \end{pmatrix}$$

$$\int_{t_0}^{t} \Phi^{-1}(s)\vec{F}(s)\,ds = \int_{1}^{t} \begin{pmatrix} 1/s \\ 0 \end{pmatrix} ds = \begin{pmatrix} \ln s \\ 0 \end{pmatrix}\Big|_1^t$$

$$= \begin{pmatrix} \ln t \\ 0 \end{pmatrix} \Rightarrow \vec{x}_p = \begin{pmatrix} 1 & t+1 \\ 1 & t \end{pmatrix}\begin{pmatrix} \ln t \\ 0 \end{pmatrix}$$

$$\vec{x}_p = \begin{pmatrix} \ln t \\ \ln t \end{pmatrix}$$

$$\Phi^{-1}(1)\,\vec{x}(1) = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} = \vec{c}$$

$$\vec{x}(t) = \Phi\,\vec{c} + \vec{x}_p$$

$$\vec{x}(t) = \begin{pmatrix} 1 & t+1 \\ 1 & t \end{pmatrix}\begin{pmatrix} -4 \\ 3 \end{pmatrix} + \begin{pmatrix} \ln t \\ \ln t \end{pmatrix}$$

$$\boxed{\vec{x}(t) = \begin{pmatrix} 3t-1 \\ 3t-4 \end{pmatrix} + \begin{pmatrix} \ln t \\ \ln t \end{pmatrix}}$$