$\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$

$\sqrt{-3} = \sqrt{3}\sqrt{-1} = i\sqrt{3}$

$\sqrt{-25} = \sqrt{25}\sqrt{-1} = 5i$

$\sqrt{-9} = 3i$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$m = \frac{-1 \pm \sqrt{9-12}}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$m^2 + 25 = 0 \Rightarrow m^2 = -25$

$m = \pm\sqrt{-25} = \pm 5i$

* $m^2 - 25 = (m+5)(m-5)$

$y^{(4)} - y'' = m^4 - m^2 = m^2(m^2-1) \neq m^2(m-1)^2$

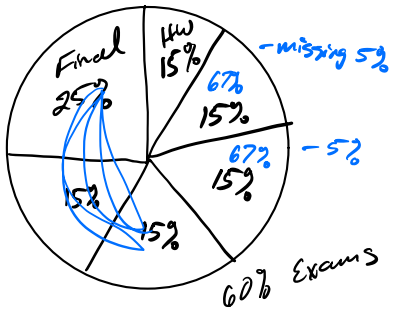
$m^2 - 2m + 1$

$m \cdot m (m+1)(m-1) = 0$

$m = 0, 0, 1, -1$

$y^{(4)} - y'' = 0$

↳ 4 soln's.



List: Needs my attention

8.2 - Homogeneous Linear Systems

Consider the system $\begin{aligned} \frac{dx}{dt} &= 2x + 2y \\ \frac{dy}{dt} &= x + 3y \end{aligned}$ which, in $\mathbf{X}' = \mathbf{AX}$ form is

$$\mathbf{X}' = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \mathbf{X}.$$

vector vector coeff. matrix

$\vec{X}' = A\vec{X}$ (2x1 = 2x2 2x1)

$\vec{X}\,A$ (2x1 2x2)

$\vec{X}' = \begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix}$, $\vec{X} = \begin{pmatrix} x \\ y \end{pmatrix}$

The solution is $\vec{X}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{t}$.

This is a superposition of $\vec{X}_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$ and $\vec{X}_2(t) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{t}$. These have the form $\vec{X}(t) = \vec{K} e^{\lambda t}$. Here, λ is an eigenvalue; it is the solution to an equation. $\vec{K}$ is an associated eigenvector.

For example $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2+2 \\ 1+3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

This has the form $A\vec{K} = \lambda \vec{K}$ $(\vec{K} \neq \vec{0})$:

Multiplying the eigenvector by A scales it by the factor λ.

Side note: $\vec{X}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{t}$.

means $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 e^{4t} \\ c_1 e^{4t} \end{pmatrix} + \begin{pmatrix} 2c_2 e^{t} \\ -c_2 e^{t} \end{pmatrix} = \begin{pmatrix} c_1 e^{4t} + 2c_2 e^{t} \\ c_1 e^{4t} - c_2 e^{t} \end{pmatrix}$

Suppose $\vec{X} = \vec{K} e^{\lambda t}$ is a solution to $\vec{X}' = A\vec{X}$. Then $\vec{X}' = \lambda \vec{K} e^{\lambda t}$

$\lambda \vec{K} e^{\lambda t} = A \vec{K} e^{\lambda t}$ $(e^{\lambda t} \neq 0) \Rightarrow \lambda \vec{K} = A\vec{K}$

$\Rightarrow A\vec{K} - \lambda \vec{K} = \vec{0} \Rightarrow (A - \lambda I)\vec{K} = \vec{0}$

We require $\vec{K} \neq \vec{0}$ (by def of eigenvector), so we seek a nontrivial solution to the system.

Recall: $A\vec{X} = \vec{0}$ with $\vec{X} \neq \vec{0} \Rightarrow \det(A) = 0$

Thus we need $\det(A - \lambda I) = 0$.

This is the characteristic equation.

Solve it to find eigenvalues λ_1, λ_2, etc.

Then for each λ, solve $(A - \lambda I)\vec{K} = \vec{0}$ to find the eigenvectors $\vec{K}_1, \vec{K}_2$, etc.

Example: Find the general solution of the given system.

$$\frac{dx}{dt} = 2x + 2y$$
$$\frac{dy}{dt} = x + 3y$$

$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \Rightarrow A - \lambda I = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

$A - \lambda I = \begin{pmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix}$

$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$6 - 5\lambda + \lambda^2 - 2 = 0 \Rightarrow \lambda^2 - 5\lambda + 4 = 0$

$(\lambda - 4)(\lambda - 1) = 0 \Rightarrow \lambda = 1, 4$

$\lambda_1 = 1$ $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\vec{K}_1 = \vec{0}$ or $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \vec{0}$

$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \vec{0} \Rightarrow k_1 + 2k_2 = 0$

$\Rightarrow k_1 + 2k_2 = 0$
$k_1 + 2k_2 = 0$

$k_1 = -2k_2$. Let $k_2 = -1$. Then $k_1 = 2$ ($k_2 = 1 \Rightarrow k_1 = -2$)

$\vec{K}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. $\vec{X}_1(t) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t$

Recall: $A - \lambda I = \begin{pmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix}$, Want $(A - \lambda I)\vec{K} = \vec{0}$

$\lambda_2 = 4$: $\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\vec{K} = \vec{0} \Rightarrow k_1 = k_2$.

Let $k_2 = 1$. Then $k_1 = 1$, so $\vec{K}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{X}_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$

$$\vec{X}(t) = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$$

Means $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2c_1 e^t + c_2 e^{4t} \\ -c_1 e^t + c_2 e^{4t} \end{pmatrix}$

$\Rightarrow$ $x(t) = 2c_1 e^t + c_2 e^{4t}$

$y(t) = -c_1 e^t + c_2 e^{4t}$

Example: Find the general solution of the given system.

$$\frac{dx}{dt} = 2x - 7y$$
$$\frac{dy}{dt} = 5x + 10y + 4z$$
$$\frac{dz}{dt} = 5y + 2z$$

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & -7 & 0 \\ 5 & 10-\lambda & 4 \\ 0 & 5 & 2-\lambda \end{vmatrix} = 0$$

Expanding along 1st row: $(2-\lambda)\begin{vmatrix} 10-\lambda & 4 \\ 5 & 2-\lambda \end{vmatrix} + 7\begin{vmatrix} 5 & 4 \\ 0 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)(\lambda^2 - 12\lambda + 20 - 20) + 7(-5\lambda + 10) = 0$

$(2-\lambda)(\lambda^2 - 12\lambda) + 35(2-\lambda) = 0$

$(2-\lambda)(\lambda^2 - 12\lambda + 35) = 0$

$(2-\lambda)(\lambda-5)(\lambda-7) = 0$

$\lambda = 2, 5, 7$

$\lambda_1 = 2$ $\quad \begin{pmatrix} 0 & -7 & 0 \\ 5 & 8 & 4 \\ 0 & 5 & 0 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \vec{0}$ $\quad R_1 \& R_3 \to k_2 = 0$

$R_2: 5k_1 + 4k_3 = 0 \Rightarrow k_1 = -\frac{4}{5}k_3$

Let $k_3 = -5 \Rightarrow k_1 = 4 \Rightarrow \vec{K}_1 = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}$

$\vec{X}_1(t) = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} e^{2t}$

$\lambda = 5$: $\begin{pmatrix} -3 & -7 & 0 \\ 5 & 5 & 4 \\ 0 & 5 & -3 \end{pmatrix}\vec{K} = \vec{0} \to \begin{pmatrix} -3 & -7 & 0 \\ 0 & -5 & 3 \\ 0 & 5 & -3 \end{pmatrix}\vec{K} = \vec{0}$

$R_2 \to 5R_1 + 3R_2$, then $R_2 \to \frac{1}{4}R_2$

$$\begin{array}{ccc} -15 & -35 & 0 \\ 15 & 15 & 12 \\ \hline 0 & -20 & 12 \end{array}$$

$$\begin{pmatrix} -3 & -7 & 0 \\ 0 & -5 & 3 \\ 0 & 0 & 0 \end{pmatrix}\vec{k} = \vec{0}$$

R_1: $-3k_1 - 7k_2 = 0$

R_2: $-5k_2 + 3k_3 = 0$

$k_1 = -\frac{7}{3}k_2$ Let $k_2 = 3$.

$k_3 = \frac{5}{3}k_2$ $k_1 = -7$, $k_3 = 5$

Using $\lambda = 7$, we would find $\vec{K}_3 = \begin{pmatrix} -7 \\ -5 \\ -5 \end{pmatrix}$

$$\vec{X}(t) = c_1\begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} -7 \\ 3 \\ 5 \end{pmatrix}e^{5t} + c_3\begin{pmatrix} -7 \\ -5 \\ -5 \end{pmatrix}e^{7t}$$

A matrix can have eigenvalues that are
- real (distinct)
- real (repeated)
- complex

If $\lambda = \alpha + \beta i$, $\bar{\lambda} = \alpha - \beta i$, and

$\vec{X}_1 = \vec{K}e^{\lambda t}$, $\vec{X}_2 = \vec{\bar{K}}e^{\bar{\lambda} t}$

By Euler's formula, $\vec{K}e^{(\alpha+\beta i)t} = \vec{K}e^{\alpha t}(\cos\beta t + \sin\beta t)$

$\vec{\bar{K}}e^{(\alpha-\beta t)i} = \vec{\bar{K}}e^{\alpha t}(\cos\beta t - \sin\beta t)$

Superposition Principle: $\vec{X}_1 = \frac{1}{2}(\vec{K}e^{\lambda t} + \vec{\bar{K}}e^{\bar{\lambda} t})$

$$= e^{\alpha t}\left[\frac{1}{2}(\vec{K}+\vec{\bar{K}})\cos\beta t - \frac{i}{2}(-\vec{K}+\vec{\bar{K}})\sin\beta t\right]$$

$\frac{1}{2}(\alpha+\beta i+\alpha-\beta i) = \alpha$ $\frac{i}{2}(-\alpha-\beta i+\alpha-\beta i) = \frac{i}{2}(-2\beta i) = \beta$

$$\vec{x}_2 = \frac{i}{2}\left(-\vec{k}e^{\lambda t} + \bar{\vec{k}}e^{\bar{\lambda} t}\right)$$

$$= e^{\alpha t}\left[\frac{i}{2}\left(-\vec{k} + \bar{\vec{k}}\right)\cos\beta t + \frac{1}{2}\left(\vec{k} + \bar{\vec{k}}\right)\sin\beta t\right]$$

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oddly placed review

System $\vec{x}' = A\vec{x}$

$\det(A - \lambda I) = 0 \rightarrow$ eigenvalues (λ)

$(A - \lambda I)\vec{k} = \vec{0} \rightarrow$ eigenvectors

$\vec{k} \neq \vec{0}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{aligned} ak_1 + bk_2 &= 0 \\ ck_1 + dk_2 &= 0 \end{aligned}$$

$$\vec{x}(t) = c_1\vec{k}_1 e^{\lambda_1 t} + c_2\vec{k}_2 e^{\lambda_2 t}$$

Theorem: Let $\lambda = \alpha + i\beta$ be an eigenvalue of $\mathbf{A}$ and let $\mathbf{B}_1 = \text{Re}(\mathbf{K})$, $\mathbf{B}_2 = \text{Im}(\mathbf{K})$. Then

$$\mathbf{X}_1 = e^{\alpha t}(\mathbf{B}_1 \cos \beta t - \mathbf{B}_2 \sin \beta t)$$
$$\mathbf{X}_2 = e^{\alpha t}(\mathbf{B}_2 \cos \beta t + \mathbf{B}_1 \sin \beta t)$$

Example: Find the general solution of the given system.

$$\frac{dx}{dt} = 4x + 5y$$
$$\frac{dy}{dt} = -2x + 6y$$

$$\begin{vmatrix} 4-\lambda & 5 \\ -2 & 6-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 10\lambda + 24 + 10 = 0$$

$$\lambda^2 - 10\lambda + 25 = -34 + 25$$

$$(\lambda - 5)^2 = -9 \Rightarrow \lambda - 5 = \pm 3i$$

$$\lambda = 5 \pm 3i$$

$\lambda = 5+3i$ $\quad \begin{pmatrix} 4-(5+3i) & 5 \\ -2 & 6-(5+3i) \end{pmatrix} \vec{k} = \vec{0}$

$$\begin{pmatrix} -1-3i & 5 \\ -2 & 1-3i \end{pmatrix} \vec{k} = \vec{0}$$

$2R_1 + (-1-3i)R_2$ OR confirm $\det(A - \lambda I) = 0$

After confirming, we can use R_2:

$$-2k_1 + (1-3i)k_2 = 0$$

$$k_2 = \frac{2}{1-3i} k_1 \quad \text{Let } k_1 = 1-3i$$

Then $k_2 = 2$.

$$\vec{K} = \begin{pmatrix} 1 - 3i \\ 2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\vec{B}_1} + \underbrace{\begin{pmatrix} -3 \\ 0 \end{pmatrix} i}_{\vec{B}_2}$$

Since

$$\mathbf{X}_1 = e^{\alpha t}(\mathbf{B}_1 \cos \beta t - \mathbf{B}_2 \sin \beta t)$$

$$\mathbf{X}_2 = e^{\alpha t}(\mathbf{B}_2 \cos \beta t + \mathbf{B}_1 \sin \beta t)$$

with $\lambda = 5 + 3i$, we have

$$\vec{X} = c_1 e^{5t}\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\cos 3t - \begin{pmatrix} -3 \\ 0 \end{pmatrix}\sin 3t\right] + c_2 e^{5t}\left[\begin{pmatrix} -3 \\ 0 \end{pmatrix}\cos 3t + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\sin 3t\right]$$

$$\vec{X} = c_1 \vec{X}_1 + c_2 \vec{X}_2$$

$$x(t) = c_1 e^{5t}(\cos 3t + 3\sin 3t) + c_2 e^{5t}(-3\cos 3t + \sin 3t)$$

$$y(t) = 2c_1 e^{5t}\cos 3t + 2c_2 e^{5t}\sin 3t$$

Repeated eigenvalues:

Example: Find the general solution of the given system.

$$\mathbf{X}' = \begin{pmatrix} 12 & -9 \\ 4 & 0 \end{pmatrix} \mathbf{X}$$

$$\begin{vmatrix} 12-\lambda & -9 \\ 4 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 12\lambda + 36 = 0$$

$(\lambda - 6)^2 = 0 \Rightarrow \lambda = 6$ (multiplicity 2)

$\lambda = 6$ $\begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix} \vec{k} = \vec{0} \Rightarrow \begin{pmatrix} 2 & -3 \\ 0 & 0 \end{pmatrix} \vec{k} = \vec{0}$

$2k_1 - 3k_2 = 0 \Rightarrow k_1 = \frac{3}{2}k_2$

Let $k_2 = 2 \Rightarrow k_1 = 3 \Rightarrow \vec{K} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\vec{X}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{6t}$ We have only one eigenvector.

In this case, we find a vector $\vec{P}$ such that $(A - \lambda I)\vec{P} = \vec{K}$.

(Rationale will come later.)

$$\begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \rightarrow \left(\begin{array}{cc|c} 6 & -9 & 3 \\ 4 & -6 & 2 \end{array}\right)$$

$R_2 \rightarrow \frac{1}{3}R_1 - \frac{1}{2}R_2$ $\begin{matrix} 2 & -3 & 1 \\ 0 & 0 & 0 \end{matrix}$, $R_1 \rightarrow \frac{1}{3}R_1$

$$\left(\begin{array}{cc|c} 2 & -3 & 1 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow 2p_1 - 3p_2 = 1 \quad \text{Let } p_2 = 0$$

$\Rightarrow p_1 = \frac{1}{2} \Rightarrow \vec{P} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$

$\vec{X}_2 = \vec{K}te^{\lambda t} + \vec{P}e^{\lambda t}$

$\vec{X}(t) = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{6t} + c_2 \left[\begin{pmatrix} 3 \\ 2 \end{pmatrix} te^{6t} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^{6t} \right]$

Example: Find the general solution of the given system.

$$\frac{dx}{dt} = 3x + 2y + 4z$$
$$\frac{dy}{dt} = 2x + 2z$$
$$\frac{dz}{dt} = 4x + 2y + 3z$$

$$\begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\begin{vmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 4 & 3-\lambda \end{vmatrix} + 4\begin{vmatrix} 2 & -\lambda \\ 4 & 2 \end{vmatrix} = 0$$

$$(3-\lambda)(\lambda^2 - 3\lambda - 4) - 2(6 - 2\lambda - 8) + 4(4 + 4\lambda) = 0$$

$$(3-\lambda)(\lambda-4)(\lambda+1) + 4(\lambda+1) + 16(\lambda+1) = 0$$

$$(\lambda+1)(-\lambda^2 + 7\lambda - 12 + 4 + 16) = 0$$

$$-(\lambda+1)(\lambda^2 - 7\lambda - 8) = 0 \Rightarrow -(\lambda+1)^2(\lambda-8) = 0$$

$\lambda = 8, -1$ (mult. 2)

$\lambda_1 = 8 \Rightarrow \vec{K}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$\lambda_2 = -1$: $\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}\vec{K} = \vec{0} \rightarrow \begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\vec{K} = \vec{0}$

$2k_1 + k_2 + 2k_3 = 0$

Let $k_3 = 0$. Then $k_1 = -\frac{1}{2}k_2$. Let $k_2 = -2$

$k_1 = 1 \Rightarrow \vec{K}_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$

↳ Let $k_2 = 0$. Then $k_1 = -k_3 \Rightarrow \vec{K}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

~~Let $k_1 = 0$. Then $k_2 = -2k_3$. Let $k_3 = 1$~~

~~Then $\vec{K}_4 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$.~~ redundant

$$\vec{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{8t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t}$$

Aforementioned Rationale:

In the case of a repeated eigenvalue but not enough eigenvectors (defective)

We have

$$\vec{x}_1 = \vec{K} e^{\lambda t}$$

$$\vec{x}_2 = \vec{K} t e^{\lambda t} + \vec{P} e^{\lambda t}$$

$$\vec{x}_3 = \frac{1}{2!} \vec{K} t^2 e^{\lambda t} + \vec{P} t e^{\lambda t} + \vec{Q} e^{\lambda t}$$

use $\vec{K}$ to find $\vec{P}$: $(A - \lambda I)\vec{P} = \vec{K}$

$(A - \lambda I)\vec{Q} = \vec{P}$

Suppose $\vec{x} = \vec{K} t e^{\lambda t} + \vec{P} e^{\lambda t}$

Then $\vec{x}' = \vec{K} e^{\lambda t} + \vec{K} \lambda t e^{\lambda t} + \vec{P} \lambda e^{\lambda t}$

So $\vec{x}' = A\vec{x}$ becomes

$$\vec{k}e^{\lambda t}+\vec{k}\lambda t e^{\lambda t}+\vec{p}\lambda e^{\lambda t}=A\left(\vec{k}te^{\lambda t}+\vec{p}e^{\lambda t}\right)$$

$$\vec{k}\lambda te^{\lambda t}=A\vec{k}te^{\lambda t} \Rightarrow \lambda\vec{k}=A\vec{k}$$

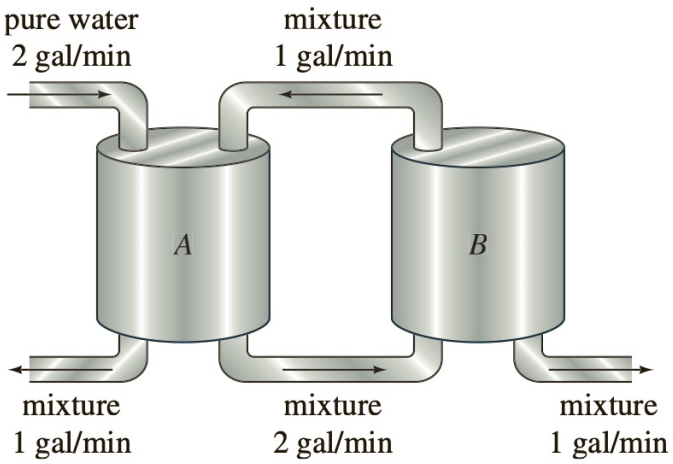
$$\Rightarrow (A-\lambda I)\vec{k}=\vec{0}$$

we knew this.

$$\vec{k}+\vec{p}\lambda=A\vec{p} \Rightarrow A\vec{p}-\vec{p}\lambda=\vec{k}$$

$$(A-\lambda I)\vec{p}=\vec{k}$$

As shown in the diagram, two large connected mixing tanks A and B initially contain 100 gallons of brine. Liquid is pumped in and out of the tanks as indicated in the figure; the mixture pumped between and out of a tank is assumed to be well-stirred. Construct a mathematical model in the form of a linear system of first-order differential equations for the number of pounds $x_1(t)$ and $x_2(t)$ of salt in tanks A and B, respectively, at time t. Write the system in matrix form.



AI response to where this model is used: This type of coupled mixing tank model shows up in several real engineering and scientific contexts:

Environmental & Water Treatment The most direct application is **wastewater treatment plants**, where tanks are linked in series or loops to progressively

dilute or react contaminants. The exact flow-in/flow-out balance you see here models how pollutant concentrations evolve through each stage.

Pharmacokinetics (Drug Modeling) This is probably the *most important* real use. The human body is modeled as a series of "compartments" — blood plasma, tissue, organs — and a drug moves between them at known rates. The math is identical to these tank equations. Pharmacologists use this to predict:

- How long a drug stays in your system
- When blood concentration peaks
- Proper dosing intervals
- Chemical Reactors

In industrial chemistry, **continuous stirred-tank reactors (CSTRs)** are literally this setup. Reactants flow in, products flow out, and intermediate products recirculate between vessels. Petroleum refining and pharmaceutical manufacturing use multi-tank CSTR networks constantly.

Groundwater Contamination Aquifers are modeled as interconnected "cells" exchanging water at known rates. When a contaminant spills, hydrologists use these equations to predict how the plume spreads between underground zones over time.

HVAC and Atmospheric Modeling The atmosphere itself is divided into compartments (troposphere, boundary layer, urban airshed) with pollutants like CO_2 or smog transferring between them — same mathematical structure.

The reason this model is so powerful is that **any system where a substance moves between well-mixed reservoirs at proportional rates** reduces to the same form of coupled linear ODEs. The tanks are just the most intuitive way to introduce the concept.

Ex:

$$\vec{X}' = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix} \vec{X}$$

$$\begin{vmatrix} 2-\lambda & 1 & 6 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)^3 = 0$$

$$\lambda = 2 \text{ (mult. 3)}$$

$A - \lambda I$

$$\begin{pmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix} \vec{K} = \vec{0} \Rightarrow \vec{K} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Use $\vec{K}$ to find $\vec{P}$: $(A - \lambda I)\vec{P} = \vec{K}$

$$\left(\begin{array}{ccc|c} 0 & 1 & 6 & 1 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad R_2 \rightarrow P_3 = 0$$

$$R_1 \rightarrow P_2 = 1$$

let $P_1 = 0 \Rightarrow \vec{P} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Use $\vec{P}$ to find $\vec{Q}$: $(A - \lambda I)\vec{Q} = \vec{P}$

$$\left(\begin{array}{ccc|c} 0 & 1 & 6 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad R_2 \to 5q_3 = 1 \Rightarrow q_3 = \frac{1}{5}$$

$$R_1: q_2 + \frac{6}{5} = 0 \Rightarrow q_2 = -\frac{6}{5}$$

let $q_1 = 0$

$$\vec{Q} = \begin{pmatrix} 0 \\ -6/5 \\ 1/5 \end{pmatrix}$$

$$\vec{X}(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t e^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t} \right]$$

$$+ c_3 \left[\frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t^2 e^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t e^{2t} + \begin{pmatrix} 0 \\ -6/5 \\ 1/5 \end{pmatrix} e^{2t} \right]$$