8.1 - Preliminary Theory–Linear Systems

Parallels exist between concepts, theorems, and vocabulary of linear differential equations and linear systems of differential equations:

- Existence and uniqueness of solutions
- The superposition principle
- Linear dependence/independence and how they relate to the Wronskian (though the Wronskian is different here)
- Fundamental set of solutions
- Homogeneous and nonhomogeneous systems
- Complementary and particular solutions

Recall from algebra: A system of equations can be represented as a matrix equation. For instance, the system

$$\begin{cases} 2x + 3y - 5z = 12 \\ 7x + 4y + 6z = 18 \\ -2x \quad + \quad z = -27 \end{cases}$$ can be represented by the matrix equation

$$\begin{pmatrix} 2 & 3 & -5 \\ 7 & 4 & 6 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 18 \\ -27 \end{pmatrix}$$

$$\begin{pmatrix} 2x + 3y - 5z \\ 7x + 4y + 6z \\ -2x \quad + z \end{pmatrix} = \begin{pmatrix} 12 \\ 18 \\ -27 \end{pmatrix}$$

In this chapter, we will primarily be concerned with linear systems of DEs that have the forms

$\mathbf{X}' = \mathbf{AX}$ and $\mathbf{X}' = \mathbf{AX} + \mathbf{F}$

homogeneous nonhomogeneous

Example: Write the given linear system in matrix form.

$$\begin{cases} \dfrac{dx}{dt} = 4x - 7y \\ \dfrac{dy}{dt} = 5x \end{cases}$$

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 4 & -7 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \vec{X}' = \begin{pmatrix} 4 & -7 \\ 5 & 0 \end{pmatrix} \vec{X}$$

Form: $\vec{X}' = A\vec{X}$

Example: Write the given linear system in matrix form.

$$\begin{cases} \dfrac{dx}{dt} = -3x + 4y + e^{-t}\sin 2t \\ \dfrac{dy}{dt} = 5x + 9z + 4e^{-t}\cos 2t \\ \dfrac{dz}{dt} = y + 6z - e^{-t} \end{cases}$$

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} -3 & 4 & 0 \\ 5 & 0 & 9 \\ 0 & 1 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} e^{-t}\sin 2t \\ 4e^{-t}\cos 2t \\ -e^{-t} \end{pmatrix}$$

Form: $\vec{X}' = A\vec{X} + \vec{F}(t)$

$$\vec{X}' = \begin{pmatrix} -3 & 4 & 0 \\ 5 & 0 & 9 \\ 0 & 1 & 6 \end{pmatrix} \vec{X} + \begin{pmatrix} \sin 2t \\ 4\cos 2t \\ -1 \end{pmatrix} e^{-t}$$

Example: Write the given linear system without the use of matrices.

$$\mathbf{X}' = \begin{pmatrix} 7 & 5 & -9 \\ 4 & 1 & 1 \\ 0 & -2 & 3 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} e^{5t} - \begin{pmatrix} 8 \\ 0 \\ 3 \end{pmatrix} e^{-2t}$$

Recall: $\frac{dx}{dt} = x'(t) = D_x$

$$\frac{dx}{dt} = 7x + 5y - 9z - 8e^{-2t}$$
$$\frac{dy}{dt} = 4x + y + z + 2e^{5t}$$
$$\frac{dz}{dt} = -2y + 3z + e^{5t} - 3e^{-2t}$$

Example: Verify that the vector $\mathbf{X}$ is a solution of the given homogeneous linear system.

$$\mathbf{X}' = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{X}; \qquad \mathbf{X} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^t + \begin{pmatrix} 4 \\ -4 \end{pmatrix} te^t$$

$$\vec{x} = \begin{pmatrix} e^t + 4te^t \\ 3e^t - 4te^t \end{pmatrix}$$

LHS

$$\Rightarrow \vec{x}'(t) = \begin{pmatrix} e^t + 4e^t(t+1) \\ 3e^t - 4e^t(t+1) \end{pmatrix} = \begin{pmatrix} 4te^t + 5e^t \\ -4te^t - e^t \end{pmatrix}$$

OR $\vec{x}' = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^t + \begin{pmatrix} 4 \\ -4 \end{pmatrix}(e^t + te^t)$

RHS $A\vec{x} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} e^t + 4te^t \\ 3e^t - 4te^t \end{pmatrix}$

$$= \begin{pmatrix} 2(e^t + 4te^t) + 1(3e^t - 4te^t) \\ -1(e^t + 4te^t) + 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4te^t + 5e^t \\ -4te^t - e^t \end{pmatrix}$$

LHS = RHS ✓

Some matrix preliminaries

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\vec{x}(t) = \begin{pmatrix} x \\ y \end{pmatrix}$

Other notation: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\rightarrow A\vec{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

$\vec{x}A = \begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Not possible

Identity matrix: $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$AI = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = IA$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$A\vec{X} = \vec{0}$ $\qquad \vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

↑ system of equations

$\vec{X}$ is the solution to the system

Linear algebra result: If $A\vec{X} = \vec{0}$ has a solution when $\vec{X} \neq \vec{0}$, then $\det(A) = 0$. ($\vec{X} = \vec{0}$ is the trivial solution.)

(associated with a nontrivial solution.

Make sure you're comfortable computing 2x2 and 3x3 determinants.

Augmented matrices: (Gaussian elimination)

① $2x + 2y = 4$
② $8x + 3y = 5$

$\rightarrow \left(\begin{array}{cc|c} 2 & 2 & 4 \\ 8 & 3 & 5 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 2 & 2 & 4 \\ 0 & -5 & -11 \end{array}\right)$

$② - 4①$ $8x+3y=5$ $-8x-8y=-16$

$-5y=-11$

$R_2 - 4R_1$

$8\ 3\ 5$

$-8\ -8\ -16$

$0\ -5\ -11$

① $2x+3y=0$

② $-4x-6y=0$

$\rightarrow \begin{pmatrix} 2 & 3 & | & 0 \\ -4 & -6 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 3 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix}$

$R_2 \rightarrow R_2 + 2R_1$

$-4\ -6\ 0$

$4\ 6\ 0$

$0\ 0\ 0$

$\begin{vmatrix} 2 & 3 \\ -4 & -6 \end{vmatrix} = -12 - (-12) = 0$