6.3 - Solutions About Singular Points

A singular point for the equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ $(*)$ is a point $x = x_0$ such that $a_2(x_0) = 0$. For our work with solving differential equations, we will only consider $x_0 = 0$. Standard form for $(*)$ is $y'' + P(x)y' + Q(x)y = 0$. Multiplying by x^2 (or by $(x - x_0)^2$ if $x_0 \neq 0$) yields $x^2y'' + x^2P(x)y' + x^2Q(x)y = 0$. By relabeling $p(x) = xP(x)$, $q(x) = x^2Q(x)$, we get $x^2y'' + xp(x)y' + q(x)y = 0$.

Definition: $x = x_0$ is a **regular singular point** if $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2Q(x)$ are analytic at $x = x_0$ and an **irregular singular point** if either $p(x)$ or $q(x)$ is not analytic at $x = x_0$.

Example: Determine the singular points of the given differential equation. Classify each singular point as regular or irregular.

$(x^2 - 9)^2y'' + (x + 3)y' + 2y = 0$

$x = 3, -3$ are singular points

Std form: $y'' + \frac{1}{(x+3)(x-3)^2}y' + \frac{2}{(x+3)^2(x-3)^2}y = 0$

$P(x) = \frac{1}{(x+3)(x-3)^2}$, $Q(x) = \frac{2}{(x+3)^2(x-3)^2}$

$x = 3$ (regular or irregular)

Examine $(x-3)P(x)$ and $(x-3)^2Q(x)$

$(x-3)P(x) = \frac{1}{(x+3)(x-3)}$ $(x-3)^2Q(x) = \frac{2}{(x+3)^2}$

not analytic at $x = 3$ $\Rightarrow$ $x = 3$ is an irregular singular point

$x = -3$ $(x+3)P(x) = \frac{1}{(x-3)^2}$ $(x+3)^2Q(x) = \frac{2}{(x-3)^2}$

$\Rightarrow$ these are both analytic at $x = -3$,

so $x = -3$ is a regular singular point

Theorem: (Frobenius' Theorem) If $x = x_0$ is a regular singular point of (∗), then there is at least one solution of the form
$y = (x-x_0)^r \sum_{n=0}^{\infty} c_n(x-x_0)^n = \sum_{n=0}^{\infty} c_n(x-x_0)^{n+r}$, where the number r is a constant to be determined.

Example: $x = 0$ is a regular singular point of the given differential equation. Show that the indicial roots of the singularity do not differ by an integer. Use the method of Frobenius to obtain two linearly independent solutions about $x = 0$. Form the general solution on $(0, \infty)$.

$y'' + \frac{5}{2x} y' + \frac{1}{2x} y = 0$

$2xy'' + 5y' + xy = 0$

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} = x^r \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

$$2x \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} + 5 \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} + x \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) c_n x^{n+r-1} + \sum_{n=0}^{\infty} 5(n+r) c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r+1} = 0$$

$$\sum_{k=0}^{\infty} [2(k+r)(k+r-1) + 5(k+r)] c_k x^{k+r-1} + \sum_{k=2}^{\infty} c_{k-2} x^{k+r-1} = 0$$

$k=0$ $\qquad$ $k=1$

$$[2r(r-1)+5r] c_0 x^{r-1} + [2(r+1)(r)+5(r+1)] c_1 x^r$$

$$+\sum_{k=2}^{\infty}\left\{\left[2(k+r)(k+r-1)+5(k+r)\right]c_k + c_{k-2}\right\}x^{k+r-1} = 0$$

$(k+r)(2k+2r-2+5) = (k+r)(2k+2r+3)$

$$\left[2r(r-1)+5r\right]c_0 = 0 \quad , \quad \left[2r(r+1)+5(r+1)\right]c_1 = 0$$

$\Rightarrow 2r^2-2r+5r = 0$ ← indicial equation

$r(2r+3) = 0$

$2r^2+3r = 0$

$c_1 = 0$

$r = 0, -\frac{3}{2}$ indicial roots (don't differ by an integer)

$$c_k = -\frac{1}{(k+r)(2k+2r+3)}c_{k-2}, \quad k = 2, 3, 4, \ldots$$

$r = 0$

$$c_k = -\frac{1}{k(2k+3)}c_{k-2}$$

$k=2 \quad c_2 = -\frac{1}{14}c_0$

$k=3 \quad c_3 = -\frac{1}{27}c_1 = 0$

$k=4 \quad c_4 = -\frac{1}{44}c_2 = \frac{1}{616}c_0$

$r = -\frac{3}{2}$

$$c_k = -\frac{1}{(k-\frac{3}{2})(2k)}c_{k-2}$$

$$c_k = -\frac{1}{k(2k-3)}c_{k-2}$$

$c_2 = -\frac{1}{2}c_0$

$c_3 = 0$

$c_4 = -\frac{1}{20}c_2 = \frac{1}{40}c_0$

Let $c_1 = c_0$ ↓

Let $c_2 = c_0$ ↓

$$y = c_0 - \frac{1}{14}c_0x^2 + \frac{1}{616}c_0x^4 + \cdots + x^{-3/2}\left(c_2 - \frac{1}{2}c_2x^2 + \frac{1}{40}c_2x^4 + \cdots\right)$$

$$y_1 = 1 - \frac{1}{14}x^2 + \frac{1}{616}x^4 + \cdots \quad , \quad y_2 = x^{-3/2}\left(1 - \frac{1}{2}x^2 + \frac{1}{40}x^4 + \cdots\right)$$

The equation $r(2r + 3) = 0$ is an indicial equation and can be generalized as follows:

$x=0$ is a regular singular point of $x^2y'' + xp(x)y' + q(x)y = 0$ if $p = xP(x)$ and $q = x^2Q(x)$ are analytic at $x=0$. That is, if $p(x) = a_0 + a_1x + \cdots$, $q(x) = b_0 + b_1x + \cdots$

Substitute the Frobenius series

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r} + (a_0 + a_1x + \cdots)\sum_{n=0}^{\infty}(n+r)c_nx^{n+r} + (b_0 + b_1x + \cdots)\sum_{n=0}^{\infty}c_nx^{n+r} = 0$$

We want the term with the smallest power on x.

$n=0$: $r(r-1) + a_0r + b_0 = 0$,

where a_0 is the constant term of $p(x) = xP(x)$ and b_0 is the constant term of $q(x) = x^2Q(x)$.

Consider $2xy'' + 5y' + xy = 0$

multiply by $\frac{x}{2}$ to get the form

$$x^2y'' + xp(x)y' + q(x)y = 0$$

$xp(x) = \frac{5}{2}x$

$$x^2y'' + \frac{5}{2}xy' + \frac{1}{2}x^2y = 0$$

$p(x) = \frac{5}{2} \Rightarrow a_0 = \frac{5}{2}$ $\quad q(x) = \frac{1}{2}x^2 \Rightarrow b_0 = 0$

$\frac{1}{2}x^2 + 0x + 0$

$$r(r-1) + \frac{5}{2}r + 0 = 0$$

$$2r(r-1) + 5r = 0 \Rightarrow 2r^2 + 3r = 0$$

$$\Rightarrow r(2r+3) = 0$$

$$r = 0, -\frac{3}{2}$$

Example: Use the method of Frobenius to obtain two linearly independent solutions about the regular singular point $x = 0$.

$9x^2y'' + 9x^2y' + 2y = 0$

Indicial roots: $x^2y'' + x^2y' + \frac{2}{9}y = 0$

$xp(x) = x^2 \Rightarrow p(x) = x + 0$, $q(x) = \frac{2}{9} = b_0$

$a_0 = 0$

$r(r-1) + a_0 r + b_0 = 0$

$r(r-1) + \frac{2}{9} = 0 \Rightarrow 9r^2 - 9r + 2 = 0$

$(3r-1)(3r-2) = 0$

$r = \frac{1}{3}, \frac{2}{3}$ (do not differ by an integer)

$$\sum_{n=0}^{\infty} 9(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty} 9(n+r)c_n x^{n+r+1} + \sum_{n=0}^{\infty} c_n 2x^{n+r} = 0$$

- Exponents

$$\sum_{k=0}^{\infty} 9(k+r)(k+r-1)c_k x^{k+r} + \sum_{k=1}^{\infty} 9(k+r-1)c_{k-1} x^{k+r}$$

$$+ \sum_{k=0}^{\infty} 2 c_k x^{k+r} = 0$$

$k=0$: $[9r(r-1)+2]c_0 x^r = 0 \Rightarrow 9r^2 - 9r + 2 = 0$,

as expected. we have $r = \frac{1}{3}, \frac{2}{3}$

$$9(k+r)(k+r-1)c_k + 9(k+r-1)c_{k-1} + 2c_k = 0$$

$$[9(k+r)(k+r-1) + 2]c_k + 9(k+r-1)c_{k-1} = 0$$

$$c_k = -\frac{9(k+r-1)}{9(k+r)(k+r-1)+2} c_{k-1}, \quad k = 1, 2, 3 \ldots$$

$r = \frac{1}{3}$

$$c_k = -\frac{9(k-2/3)}{9(k+1/3)(k-2/3)+2}\,c_{k-1}$$

$$c_k = -\frac{3(3k-2)}{(3k+1)(3k-2)+2}\,c_{k-1}$$

$$c_k = -\frac{3(3k-2)}{9k^2-3k}\,c_{k-1}$$

$$c_k = -\frac{3k-2}{k(3k-1)}\,c_{k-1}$$

$r = \frac{2}{3}$

$$c_k = -\frac{9(k-1/3)}{9(k+2/3)(k-1/3)+2}\,c_{k-1}$$

$$c_k = -\frac{3(3k-1)}{(3k+2)(3k-1)+2}\,c_{k-1}$$

$$c_k = -\frac{3(3k-1)}{9k^2+3k}\,c_{k-1}$$

$$c_k = -\frac{3k-1}{k(3k+1)}\,c_{k-1}$$

$k=1$ $\quad c_1 = -\frac{1}{2}c_0$ $\qquad c_1 = -\frac{2}{4}c_0 = -\frac{1}{2}c_0$

$k=2$ $\quad c_2 = -\frac{4}{10}c_1 = \frac{1}{5}c_0$ $\qquad c_2 = -\frac{5}{14}c_1 = \frac{5}{28}c_0$

we get $x^{1/3}\left(c_0 - \frac{1}{2}c_0x + \frac{1}{5}c_0x^2 + \cdots\right)$ Let $c_1 = c_0$

we get $x^{2/3}\left(c_0 - \frac{1}{2}c_0x + \frac{5}{28}c_0x^2 + \cdots\right)$ Let $c_2 = c_0$

x^r

General solution

$$y = c_1x^{1/3}\left(1 - \frac{1}{2}x + \frac{1}{5}x^2 + \cdots\right) + c_2x^{2/3}\left(1 - \frac{1}{2}x + \frac{5}{28}x^2 + \cdots\right)$$

Two linearly independent solutions

$$y_1 = x^{1/3}\left(1 - \frac{1}{2}x + \frac{1}{5}x^2 + \cdots\right),\quad y_2 = x^{2/3}\left(1 - \frac{1}{2}x + \frac{5}{28}x^2 + \cdots\right)$$

If r_1 and r_2 differ by an integer–that is, if $r_2 - r_1 \in \mathbb{Z}$–we can use r_1 to find y_1 and then reduction of order to find r_2.

Example: $x = 0$ is a regular singular point of the given differential equation. Show that the indicial roots of the singularity differ by an integer. Use the method of Frobenius to obtain at least one series solution about $x = 0$. Use reduction of order and a CAS to find a second solution. Form the general solution on $(0, \infty)$.

$y'' + \frac{3}{x}y' - 2y = 0$

$x^2y'' + 3xy' - 2x^2y = 0 \Rightarrow a_0 = 3, b_0 = 0$

$r(r-1) + 3r = 0 \Rightarrow r^2 - r + 3r = 0$

$r^2 + 2r = 0 \Rightarrow r = 0, -2$, which differ by an integer

we will find that $c_k = \frac{2}{(k+r)(k+r+2)} c_{k-2}$,

$k = 2, 3, 4, \ldots$

and $c_1 = 0$

$r=0$

$$c_k = \frac{2}{k(k+2)} c_{k-2}$$

$k=2$ $\quad c_2 = \frac{1}{4} c_0$

$k=3$ $\quad c_3 = \frac{2}{15} c_1 = 0$

$k=4$ $\quad c_4 = \frac{1}{48} c_0$

$r=-2$

$$c_k = \frac{2}{k(k-2)} c_{k-2}$$

undefined at $k=2$.

We get no terms from this

The "at least one" power series solution is

$$y_1 = 1 + \frac{1}{4}x^2 + \frac{1}{48}x^4 + \cdots$$

To find y_2, we use reduction of order.

Recall: $y'' + P(x)y' + Q(x)y = 0$

with known solution y_1,

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

Here, $P(x) = \frac{3}{x} \Rightarrow e^{-\int P(x)dx} = e^{-\int \frac{3}{x}dx}$

$$= x^{-3}$$

$$y_2 = y_1 \int \frac{1}{x^3 (1 + \frac{1}{4}x^2 + \frac{1}{48}x^4 + \cdots)^2} dx$$

Wolf