6.2 - Solutions About Ordinary Points

Consider the differential equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, which in standard form is $y'' + P(x)y' + Q(x)y = 0$.

Definition: A point $x = x_0$ is an **ordinary point** if P and Q are analytic at x_0 (meaning power series representations centered at $x = x_0$) exist. A point that is not ordinary is **singular**.

Theorem: If $x = x_0$ is an ordinary point of $a_2(x)y''+a_1(x)y'+a_0(x)y = 0$, then we can always find two linearly independent solutions of the form $y = \sum_{n=0}^{\infty} c_n(x - x_0)^n$.

Example: Find two power series solutions of the given differential equation about the ordinary point $x = 0$.

$y'' + 2xy' + 2y = 0$

$y = \sum_{n=0}^{\infty} c_n x^n$, $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} 2nc_n x^{n-1} + \sum_{n=0}^{\infty} 2c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} 2nc_n x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0$$

- Exponents

$$\sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k + \sum_{k=1}^{\infty} 2kc_k x^k + \sum_{k=0}^{\infty} 2c_k x^k = 0$$

- Indices

$k=0$ $k=0$

$$2c_2 + 2c_0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)c_{k+2} + (2k+2)c_k\right] x^k = 0$$

Constant terms:

$2c_2 + 2c_0 = 0 \Rightarrow c_2 = -c_0$

From the sigma, $(k+2)(k+1)c_{k+2} + 2(k+1)c_k = 0$,

$k = 1, 2, 3, \cdots$

Recurrence relation

$$c_{k+2} = -\frac{2}{k+2} c_k, \quad k = 1, 2, 3, \cdots$$

$k=1$ $\quad c_3 = -\frac{2}{3} c_1$

$k=2$ $\quad c_4 = -\frac{1}{2} c_2 = \frac{1}{2} c_0$

$k=3$ $\quad c_5 = -\frac{2}{5} c_3 = \frac{4}{15} c_1$

Recall $y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$

Here, $y = c_0 + c_1 x - c_0 x^2 - \frac{2}{3} c_1 x^3 + \frac{1}{2} c_0 x^4 + \frac{4}{15} c_1 x^5 + \cdots$

$$y = c_0\left(1 - x^2 + \frac{1}{2}x^4 + \cdots\right) + c_1\left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 + \cdots\right)$$

Two solutions:

$y_1 = 1 - x^2 + \frac{1}{2}x^4 + \cdots$

$y_2 = x - \frac{2}{3}x^2 + \frac{4}{15}x^5 + \cdots$

Example: Find two power series solutions of the given differential equation about the ordinary point $x = 0$.

$xy'' + 2y''$

$(x + 2)y'' + xy' - y = 0$

$$y = \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$(x+2)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} + \sum_{n=2}^{\infty} 2n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n$$

$$- \sum_{n=0}^{\infty} c_n x^n = 0$$

$k = n-1$, $n = k+1$

$$\sum_{k=1}^{\infty} (k+1)k\, c_{k+1} x^k + \sum_{k=0}^{\infty} 2(k+2)(k+1)\, c_{k+2} x^k + \sum_{k=1}^{\infty} k c_k x^k$$

$$- \sum_{k=0}^{\infty} c_k x^k = 0$$

$k=0$ $\quad$ $k=0$

$$4c_2 - c_0 +$$

$$\sum_{k=1}^{\infty} \left[k(k+1)c_{k+1} + 2(k+1)(k+2)c_{k+2} + (k-1)c_k \right] x^k = 0$$

$$\Rightarrow c_2 = \frac{1}{4} c_0, \quad c_{k+2} = -\frac{k-1}{2(k+1)(k+2)} c_k - \frac{k}{2(k+2)} c_{k+1},$$

$k = 1, 2, 3, \ldots$

$k=1 \quad c_3 = 0c_1 - \frac{1}{6}c_2 = -\frac{1}{24}c_0$

$k=2 \quad c_4 = -\frac{1}{24}c_2 - \frac{1}{4}c_3$

$= -\frac{1}{96}c_0 + \frac{1}{96}c_0 = 0$

$k=3 \quad c_5 = -\frac{1}{20}c_3 - \frac{3}{10}c_4$

$= \frac{1}{480}c_0$

Recall: $y = c_0 + c_1 x + c_2 x^2 + c_3 x^4 + \cdots$

Here, $y = c_0 + c_1 x + \frac{1}{4}c_0 x^2 - \frac{1}{24}c_0 x^3 + \frac{1}{480}c_0 x^5 + \cdots$

$y_1 = 1 + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \cdots, \quad y_2 = x$

Example: Find two power series solutions of the given differential equation about the ordinary point $x = 0$.

$y'' - (x+1)y' - y = 0$

$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=1}^{\infty} n c_n x^{n-1}$

$- \sum_{n=0}^{\infty} c_n x^n = 0$

$\sum_{k=0} (k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} k c_k x^k - \sum_{k=0}^{\infty} (k+1)c_{k+1}x^k$

$- \sum_{k=0}^{\infty} c_k x^k = 0$

$2c_2 - c_1 - c_0 = 0 \Rightarrow c_2 = \frac{1}{2}c_0 + \frac{1}{2}c_1$

$k=0 \quad k=0 \quad k=0$

$$c_{k+2} = \frac{1}{k+2}c_k + \frac{1}{k+2}c_{k+1}, \quad k = 1, 2, 3, \ldots$$

To separate c_0 terms and c_1 terms:

Let $c_1 = 0$

$c_2 = \frac{1}{2}c_0$

$k=1$ $c_3 = \frac{1}{3}c_1 + \frac{1}{3}c_2 = \frac{1}{6}c_0$

$k=2$ $c_4 = \frac{1}{4}c_2 + \frac{1}{4}c_3$

$= \frac{1}{8}c_0 + \frac{1}{24}c_0 = \frac{1}{6}c_0$

Let $c_0 = 0$

$c_2 = \frac{1}{2}c_1$

$c_3 = \frac{1}{3}c_1 + \frac{1}{3}c_2$

$= \frac{1}{3}c_1 + \frac{1}{6}c_1 = \frac{1}{2}c_1$

$c_4 = \frac{1}{4}c_2 + \frac{1}{4}c_3$

$= \frac{1}{8}c_1 + \frac{1}{8}c_1 = \frac{1}{4}c_1$

$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$

Here $y = c_0 + c_1 x + \frac{1}{2}c_0 x^2 + \frac{1}{2}c_1 x^2 + \frac{1}{6}c_0 x^3 + \frac{1}{2}c_1 x^3 + \cdots$

$y_1 = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots, \quad y_2 = x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \cdots$

Use an appropriate Maclaurin series to find two power series solutions of the given differential equation about the ordinary point $x = 0$.

$y'' + e^x y' - y = 0$

Recall: $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \left(1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \cdots\right) \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + 30c_6 x^4 + \cdots$$

$$\left(1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \cdots\right)\left(c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots\right)$$

$$-c_0 - c_1 x - c_2 x^2 - c_3 x^3 - c_4 x^4 - \cdots = 0$$

For $e^x y'$:

$$\left(1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \cdots\right)\left(c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots\right)$$

$$c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$$

$$+ c_1 x + 2c_2 x^2 + 3c_3 x^3 + \cdots$$

$$\tfrac{1}{2}c_1 x^2 + c_2 x^3 + \tfrac{3}{2}c_3 x^4$$

$$+ \tfrac{1}{6}c_1 x^3 + \cdots$$

Constants: $2c_2 + c_1 - c_0 = 0 \Rightarrow c_2 = \frac{1}{2}c_0 - \frac{1}{2}c_1$

x terms: $6c_3 + 2c_2 + c_1 - c_1 = 0 \Rightarrow c_3 = -\frac{1}{3}c_2$

x^2 terms: $12c_4 + 3c_3 + 2c_2 + \frac{1}{2}c_1 - c_2 = 0$

$c_4 = -\frac{1}{4}c_3 - \frac{1}{12}c_2 - \frac{1}{24}c_1$ and so on

$$c_2 = \frac{1}{2}c_0 - \frac{1}{2}c_1 \quad , \quad c_3 = -\frac{1}{6}c_0 + \frac{1}{6}c_1$$

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

$$y = c_0 + c_1 x + \frac{1}{2}c_0 x^2 - \frac{1}{2}c_1 x^2 - \frac{1}{6}c_0 x^3 + \frac{1}{6}c_1 x^3 + \cdots$$

$$y_1 = 1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \cdots, \quad y_2 = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$$