6.1 - Review of Power Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

A power series has the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

In our work for solving DEs, $a = 0$, so

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$
$$= c_0 + c_1 x + \sum_{n=2}^{\infty} c_n x^n$$

Definition: A function is **analytic at a point a** if it can be represented with a power series centered at a with radius $R \neq 0$. meaning for multiple x-values

Example: Rewrite the given expression using a single power series whose general term involves x^k.

$$\sum_{n=1}^{\infty} n c_n x^{n-1} + 3\sum_{n=0}^{\infty} c_n x^{n+2} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$$
$$+ 3c_0 x^2 + 3c_1 x^3 + \cdots$$
$$= c_1 + 2c_2 x + (3c_3 + 3c_0)x^2 + (4c_4 + 3c_1)x^3 + \cdots$$

Let $k = n-1$ Then $n = k+1$

Let $k = n+2$ Then $n = k-2$

$$\sum_{k=0}^{\infty} (k+1)c_{k+1} x^k + \sum_{k=2}^{\infty} 3c_{k-2} x^k$$

$k=0$ $k=1$

$$c_1 + 2c_2 x + \sum_{k=2}^{\infty} (k+1)c_{k+1} x^k + \sum_{k=2}^{\infty} 3c_{k-2} x^k$$

$$c_1 + 2c_2 x + \sum_{k=2}^{\infty} \left[(k+1)c_{k+1} + 3c_{k-2}\right] x^k$$

Find a power series solution $y = \sum_{n=0}^{\infty} c_n x^n$ of the given linear first-order differential equation.

$4y' + y = 0$

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \cdots$$

← the index increases by 1 when we differentiate

$$4\sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

- Make exponents match
- Make indices match

$$\sum_{k=0}^{\infty} 4(k+1)c_{k+1} x^k + \sum_{k=0}^{\infty} c_k x^k = 0$$

$$\sum_{k=0}^{\infty} \left[4(k+1)c_{k+1} + c_k\right] x^k = 0$$

$$4(k+1)c_{k+1} + c_k = 0, \quad k = 0, 1, 2, \ldots$$

$$c_{k+1} = -\frac{1}{4(k+1)} c_k, \quad k = 0, 1, 2, \ldots$$

Recurrence relation

$k=0$ $\quad c_1 = -\frac{1}{4(1)} c_0$

$k=1$ $\quad c_2 = -\frac{1}{4(2)} c_1 = -\frac{1}{4(2)}\left(-\frac{1}{4(1)}\right) c_0$

$k=2$ $\quad c_3 = -\frac{1}{4(3)} c_2 = -\frac{1}{4(3)}\left(-\frac{1}{4(2)}\right)\left(-\frac{1}{4(1)}\right) c_0$

Guess $c_n = \frac{(-1)^n}{4^n n!} c_0$

So $y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$

becomes $y = c_0 - \frac{1}{4} c_0 x + \frac{1}{32} c_0 x^2 - \frac{1}{4^3 3!} c_0 x^3 + \cdots$

$y = c_0\left(1 - \frac{1}{4}x + \frac{1}{4^2 2!} x^2 - \frac{1}{4^3 3!} x^3 + \cdots\right)$

$y = c e^{-1/4 x}$ ← closed form solution

(Usually we won't find this)

Example: The given function is analytic at $a = 0$. Use appropriate series and multiplication to find the first four nonzero terms of the Maclaurin series of the given function.

$e^{-x}\cos x$

Recall: $e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \cdots$

and $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$

$e^{-x}\cos x =$

$$\left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 \cdots\right)\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots\right)$$

$$= 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots$$
$$- \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{12}x^5 + \cdots$$
$$+ \frac{1}{24}x^4 - \frac{1}{24}x^5 + \cdots$$

$$e^{-x}\cos x = 1 - x + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \cdots$$