5.2 - Linear Models: Boundary-Value Problems

5.2a

Consider $y'' + 3y = 0,\ y(0) = 0,\ y(\pi) = 0$

BVP because these #s don't match

The trivial solution $y = 0$.

We can find that $y = a\cos(\sqrt{3}x) + b\sin(\sqrt{3}x)$

Since $y(0) = 0$, $a = 0$

$y(\pi) = 0 \Rightarrow b\sin(\sqrt{3}\pi) = 0 \Rightarrow b = 0$.

We only have the trivial solution.

Now consider $y'' + 4y = 0,\ y(0) = 0,\ y(\pi) = 0$

$m^2 + 4 = 0 \Rightarrow m = \pm 2i \Rightarrow y = a\cos 2x + b\sin 2x$

$y(0) = 0 = a = 0 \Rightarrow y = b\sin 2x$

$y(\pi) = 0 \Rightarrow b \in \mathbb{R}$

$y = b\sin 2x$ is a solution to the BVP for $b \in \mathbb{R}$. The BVP has an infinite number of solutions.

lambda

Definition: A number λ for which there exist nontrivial solutions to $y'' + P(x)y' + \lambda Q(x)y = 0,\ y(a) = 0,\ y(b) = 0$ is called an **eigenvalue**, and a

solution $y = ay_1 + by_2$ that satisfies the BVP is an **eigenfunction** associated with the eigenvalue λ.

In the previous example, $P(x) = 0$, $Q(x) = 1$, $\lambda = 4$ is the eigenvalue, and $y = b\sin 2x$, $b \in \mathbb{R}$ is the eigenfunction.

$y'' + 3y = 0$ has no eigenvalue.

We generalize this:

Example: Determine eigenvalues and eigenfunctions for $y'' + \lambda y = 0,\ y(0) = 0,\ y(L) = 0\ (L > 0)$.

Case I: $\lambda = 0$. Then $y = ax + b$, and the boundary conditions give $a = b = 0$, the trivial solution – no eigenvalues here.

Case II: $\lambda < 0$. Now let $\lambda = -\alpha^2$

so $y'' - \alpha^2 y = 0 \Rightarrow m^2 - \alpha^2 = 0$

$m = \pm\alpha \Rightarrow y = ae^{\alpha x} + be^{-\alpha x}$.

Sidebar: $\cosh x = \frac{e^x + e^{-x}}{2}$, $\sinh x = \frac{e^x - e^{-x}}{2}$,

so we can write $y = a\cosh \alpha x + b\sinh \alpha x$

$y(0) = 0 \Rightarrow a = 0$, $y(L) = 0 \Rightarrow b = 0$

Again, only the trivial solution. No eigenvalue.

Case III: $\lambda > 0$. say $\lambda = \alpha^2$.

$$y'' + \alpha^2 y = 0 \Rightarrow m = \pm \alpha i$$

$$y = a\cos\alpha x + b\sin\alpha x$$

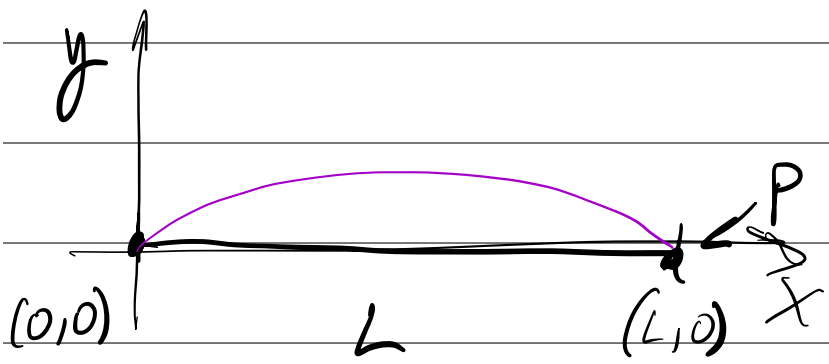
$y(0) = 0 \Rightarrow a = 0$. $y(L) = 0$ yields

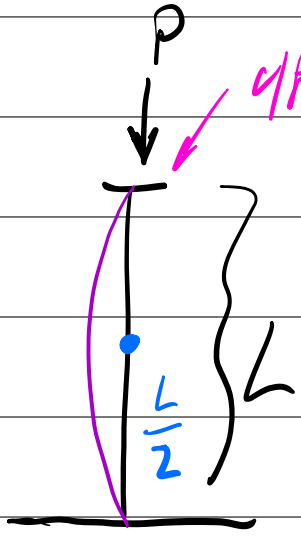
$$b\sin\alpha L = 0$$

This is true (for $b \neq 0$) whenever $\alpha L = n\pi$, $n \in \mathbb{Z}^+$. Assuming $\alpha > 0$ and $L > 0$,

$$\alpha = \frac{n\pi}{L} \Rightarrow \lambda = \alpha^2 = \frac{n^2\pi^2}{L^2}.$$

This gives an infinite sequence of eigenvalues $\lambda_n = \frac{n^2\pi^2}{L^2}$, $n = 1, 2, 3, \ldots$





5.2b

Consider a vertical beam of length L with load P applied to one end or a road with compressive force P applied to one end. Theory of elasticity provides the differential equation $EI\dfrac{d^2y}{dx^2} = -Py$ or $EI\dfrac{d^2y}{dx^2} + Py = 0$ which in standard form is

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0$$

The product EI is the flexural rigidity of the beam, E is Young's modulus of elasticity, and I is the moment of inertia of a cross-section of the beam (the shape's inherent resistance to bending).

Considering $P(x) = 0$, $Q(x) = 1$, and letting $\lambda = \dfrac{P}{EI}$, this has the form

$$y'' + \lambda y = 0$$

with this substitution and considering $y(0) = 0$, $y(L) = 0$, we have solutions $y = b\sin\left(\frac{n\pi x}{L}\right)$.

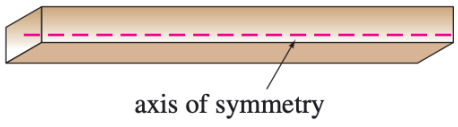
So the column/road will buckle/deflect if $P = \dfrac{n^2\pi^2 EI}{L^2}$. When $n = 1$, we have the first buckling load.

If we restrain / support the system at $L/2$, then the smallest critical (buckling) load will be $P = \dfrac{4\pi^2 EI}{L^2}$.

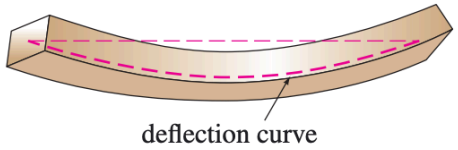
Considering a beam of length L, the bending moment (response or resistance to load) $M(x)$ is related to the load per unit length $w(x)$ by $\frac{d^2M}{dx^2} = w(x)$ (this is from theory of elasticity). Assuming down is positive, $M(x) = EI\kappa$ is proportional to the curvature of the elastic curve. From multivariable calculus, curvature is $\kappa = \frac{y''}{[1+(y')^2]^{3/2}}$. For small deflection, $y' \approx 0$, so $M = EIy''$. We see that $M'' = w(x)$ becomes

$$EI\frac{d^4y}{dx^4} = w(x)$$

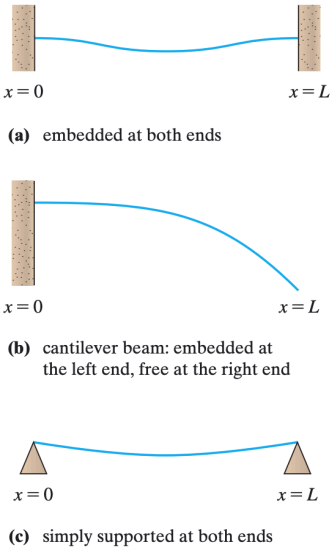
Definition: The graph of the function $y(x)$ is the **deflection curve** of the beam.



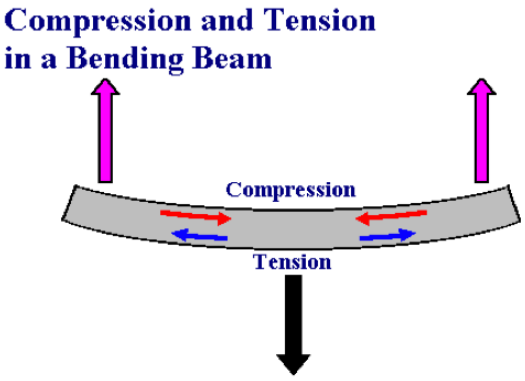
(a)



(b)



For the deflection $y(x)$, we have the following:
y' is the slope of the curve;
a nonzero multiple of y'' is the bending moment ($M = EIy''$);
y''' is the shear force. $\frac{dM}{dx} = EI\frac{d^3y}{dx^3}$



If the end of the beam is

- embedded, $y = 0, y' = 0$
- free, $y'' = 0, y''' = 0$
- simply supported / hinged, $y = 0, y'' = 0$

5.2c

Example: Solve the differential equation $EI\frac{d^4y}{dx^4} = w(x)$ subject to the appropriate boundary conditions. The beam is of length L, and w_0 is a constant.

(a) The beam is simply supported at both ends, and $w(x) = w_0, 0 < x < L$.

$y(0) = 0 \quad y(L) = 0$

$y''(0) = 0 \quad y''(L) = 0$

$$\frac{d^4y}{dx^4} = \frac{w_0}{EI}$$

$$y''' = \frac{w_0}{EI}x + C_1$$

$$y'' = \frac{w_0}{2EI}x^2 + C_1x + C_2$$

$$y' = \frac{w_0}{6EI}x^3 + \frac{C_1}{2}x^2 + C_2x + C_3$$

$$y = \frac{w_0}{24EI}x^4 + \frac{C_1}{6}x^3 + \frac{C_2}{2}x^2 + C_3x + C_4$$

$y(0) = 0 \Rightarrow C_4 = 0 \quad y''(0) = 0 \Rightarrow C_2 = 0$

$$y = \frac{w_0}{24EI}x^4 + \frac{C_1}{6}x^3 + C_3x$$

$$y(L) = 0 \Rightarrow 0 = \frac{w_0}{24EI}L^4 + \frac{C_1}{6}L^3 + C_3L$$

$$y''(L) = 0 \Rightarrow 0 = \frac{w_0}{2EI}L^2 + C_1L$$

...

We find that $\boxed{y(x) = \frac{w_0}{24EI}(x^4 - 2Lx^3 + L^3x)}$

(b) Use a graphing utility to graph the deflection curve when $w_0 = 24EI$ and $L = 1$.

Then $y(x) = x^4 - 2x^3 + x, \quad 0 \le x \le 1$

