5.1 - Linear Models: Initial-Value Problems

Undamped motion:

Example: A mass weighing 32 pounds stretches a spring 2 feet. Determine the amplitude and period of motion if the mass is initially released from a point 1 foot above the equilibrium position with an upward velocity of 2 ft/s. How many complete cycles will the mass have completed at the end of 4π seconds?

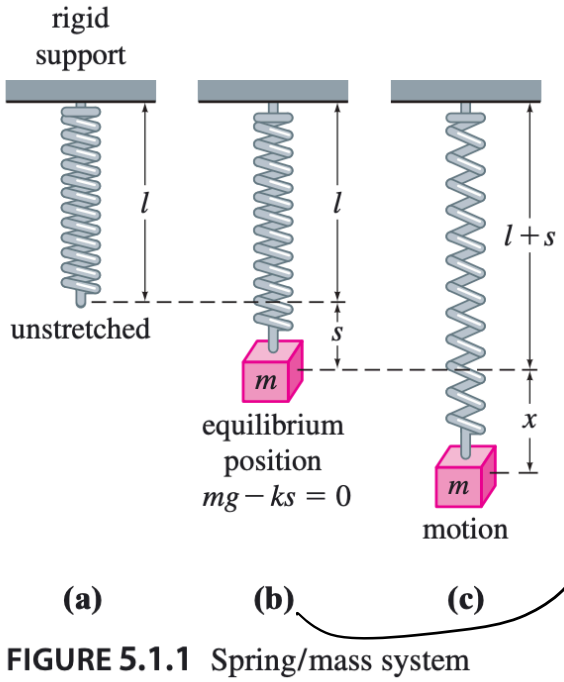


FIGURE 5.1.1 Spring/mass system

Hooke's Law

$F = k|s|$ $\quad F = 32$ lb, $|s| = 2$ ft

$32 = k \cdot 2 \Rightarrow k = 16$ lb/ft

$\Rightarrow mg = ks \Rightarrow mg - ks = 0$

Force due to gravity ↑ restoring force of the spring

Assuming down is positive, if the mass is moving, we have $F = ma$ (Newton's 2nd law) with $a = \frac{d^2x}{dt^2}$

So $m\frac{d^2x}{dt^2} = -k(x+s) + mg$

$m\frac{d^2x}{dt^2} = -kx - ks + mg$ $\quad = 0$

$$m\frac{d^2x}{dt^2} + kx = 0$$, where

$x(t)$ is distance from equilibrium, m is the mass, and k is the spring constant.

Units:

	x	k	m	g
metric	m or cm	N/m	kg	$9.8\ m/s^2$
SI	ft	lb/ft	slugs (lb/32)	$32\ ft/s^2$

mass weighs 32 lb ⇒ $32 = 32m \Rightarrow m = 1$ (lb, g, mass)

$m\frac{d^2x}{dt^2} + kx = 0$ Dividing by m and letting $\omega^2 = \frac{k}{m}$ yields

$\frac{d^2x}{dt^2} + \omega^2 x = 0 \Rightarrow$ Aux eqn. $r^2 + \omega^2 = 0$

$r = \pm \omega i$ but $\omega = \sqrt{\frac{k}{m}}$

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

For the above example, $k = 16$, $m = 1$

$\Rightarrow \omega = \sqrt{\frac{16}{1}} = 4$

$$x(t) = c_1 \cos 4t + c_2 \sin 4t$$

Released from 1 ft above equilibrium $x(0) = -1$

Upward velocity 2 ft/s $x'(0) = -2$

$x(0) = -1 \Rightarrow c_1 = -1$, $x'(t) = -4\sin 4t + 4c_2 \cos 4t$

$x'(0) = -2 \Rightarrow -2 = 4c_2 \Rightarrow c_2 = -\frac{1}{2}$

Equation of motion: $x(t) = -\cos 4t - \frac{1}{2}\sin 4t$

Period of motion $T = \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2}$ sec/cycle

frequency: $f = \frac{1}{T} = \frac{2}{\pi}$ cycles/sec (hertz)

At the end of 4π sec, 8 cycles are completed

To find the amplitude:

Consider $a\cos\theta + b\sin\theta$



$$\sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta\right)$$

Suppose $\exists\ \varphi\ \ni \frac{a}{\sqrt{a^2+b^2}} = \sin\varphi$

and $\frac{b}{\sqrt{a^2+b^2}} = \cos\varphi$

Then we have $\sqrt{a^2+b^2}\left(\sin\varphi\cos\theta + \cos\varphi\sin\theta\right)$

$= \sqrt{a^2+b^2}\ \sin(\theta+\varphi)$

So if $x(t) = c_1\cos\omega t + c_2\sin\omega t$,

then $x(t) = \sqrt{c_1^2+c_2^2}\ \sin(\omega t+\varphi)$,

where $\tan\varphi = \frac{c_1}{c_2}$

$x(t) = -\cos 4t - \frac{1}{2}\sin 4t$

$\sqrt{c_1^2+c_2^2} = \sqrt{1+\frac{1}{4}} = \frac{\sqrt{5}}{2}$

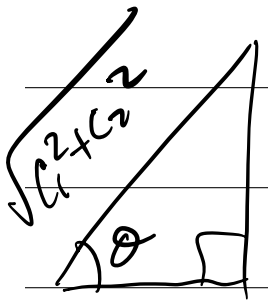
$\tan\varphi = \frac{c_1}{c_2} = 2 \Rightarrow \varphi_{ref} \approx 1.11$

but $\varphi \in$ QIII $\Rightarrow \varphi = \pi + 1.11 \approx 4.25$

$$x(t) = \frac{\sqrt{5}}{2}\sin(4t + 4.25)$$

Generalization: $m\frac{d^2x}{dt^2} + kx = 0$ yields

$x(t) = c_1 \cos\omega t + c_2 \sin\omega t$



$x(t) = \sqrt{c_1^2+c_2^2}\,\sin(\omega t+\phi)$,

where $\tan\phi = \frac{c_1}{c_2}$, $\phi_{ref} = \tan^{-1}\frac{c_1}{c_2}$

[I'll use and the signs of c_1, c_2 to identify the correct quadrant.]

$F = k|s|$ gives k (F is force - lb, Newtons), M is mass (kg, slugs), and $\omega = \sqrt{\frac{k}{m}}$.

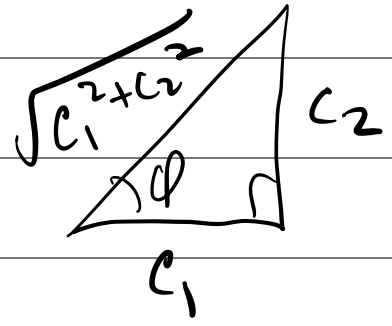
It is also possible to express the solution in terms of cosine.

$m\frac{d^2x}{dt^2} + kx = 0 \Rightarrow x(t) = c_1\cos\omega t + c_2\sin\omega t.$

$x(t) = \sqrt{c_1^2+c_2^2}\,\cos(\omega t-\phi)$,

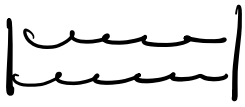
where $\tan\phi = \frac{c_2}{c_1}$.

Justification: $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$



$\tan\phi = \frac{c_2}{c_1}$

Multiple springs in parallel: $k_{\text{eff}} = k_1 + k_2$



Multiple springs in series: $k_{\text{eff}} = \dfrac{k_1 k_2}{k_1 + k_2}$



Damped motion: $m\dfrac{d^2x}{dt^2} = -kx - \beta\dfrac{dx}{dt}$

ma → restoring force from spring ← damping constant

damping force is proportional to the instantaneous velocity.

$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

Let $2\lambda = \frac{\beta}{m}$ and $\omega^2 = \frac{k}{m}$

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

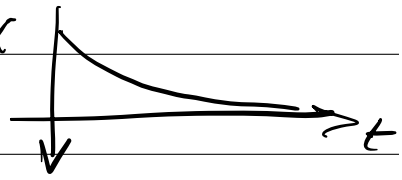
Aux equation: $r = \dfrac{-2\lambda \pm \sqrt{4\lambda^2 - 4\omega^2}}{2}$

$r = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$ (real, distinct)

Case I: $\lambda^2 - \omega^2 > 0$

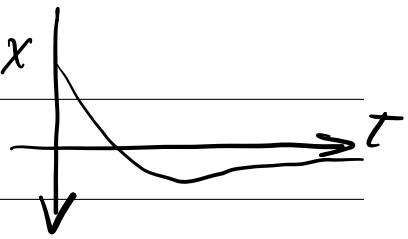
$$x(t) = c_1 e^{(-\lambda + \sqrt{\lambda^2 - \omega^2})t} + c_2 e^{(-\lambda - \sqrt{\lambda^2 - \omega^2})t}$$

This does not oscillate. This system is **overdamped.**



Case II: $\lambda^2 - \omega^2 = 0$ OR

$$x(t) = c_1 e^{-\lambda t} + c_2 t e^{-\lambda t}$$



This is **critically damped.**

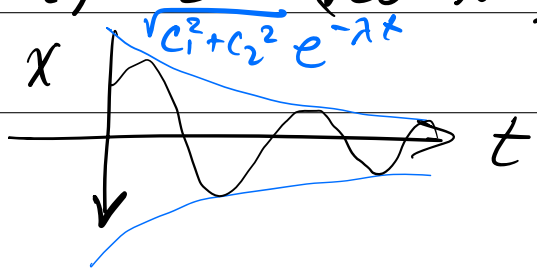
(Not gonna happen by accident.)

In case I and case II, equilibrium is crossed at most 1 time.

Case III: $\lambda^2 - \omega^2 < 0$ (complex)

$$x(t) = e^{-\lambda t}\left[c_1 \cos\left(\sqrt{\omega^2-\lambda^2}\,t\right) + c_2 \sin\left(\sqrt{\omega^2-\lambda^2}\,t\right)\right]$$

This is underdamped.



Driven motion: $m\frac{d^2x}{dt^2} + kx + \beta\frac{dx}{dt} = f(t)$

Nonhomogeneous → undetermined coefficients or variation of parameters.

LR-series circuit

LRC-series circuit: $L\frac{di}{dt} + Ri + \frac{1}{C}q = E(t)$, which is

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

$R\frac{dq}{dt} + \frac{1}{C}q$ — RC-series circuit

"underdamped," "critically damped," and "overdamped" have the same meanings as with spring/mass systems.

The complementary function is the transient term/solution, and the particular solution is the steady-state term/solution.

Example: Find the charge on the capacitor in an LRC-series circuit when $L = \frac{1}{4}$ h, $R = 20\ \Omega$, $C = \frac{1}{300}$ f, $E(t) = 0$ V, $q(0) = 4$ C, and $i(0) = 0$ A. Is the charge on the capacitor ever equal to zero?

$$\frac{1}{4}\frac{d^2q}{dt^2} + 20\frac{dq}{dt} + 300q = 0$$

$$q'' + 80q' + 1200q = 0 \Rightarrow (m+60)(m+20) = 0$$

$$m = -20, -60$$

$$q(t) = c_1 e^{-20t} + c_2 e^{-60t}$$

$$q'(t) = -20c_1 e^{-20t} - 60c_2 e^{-60t}$$

$$q(0) = 4 \Rightarrow c_1 + c_2 = 4 \qquad \rightarrow \quad c_1 + c_2 = 4$$

$$q'(0) = 0 \Rightarrow -20c_1 - 60c_2 = 0 \qquad -c_1 - 3c_2 = 0$$

$$-2c_2 = 4 \Rightarrow c_2 = -2$$

Then $c_1 = 6$. $\boxed{q(t) = 6e^{-20t} - 2e^{-60t}}$

Does $q(t) = 0$? $6e^{-20t} - 2e^{-60t} = 0$

$$3e^{40t} - 1 = 0 \Rightarrow e^{40t} = \frac{1}{3}$$

$$t = \frac{1}{40}\ln\frac{1}{3} < 0.$$

It's never zero for $t > 0$.