Differential operator: For $y = y(x)$

$Dy = \frac{dy}{dx}$ and $D^2 y = \frac{d^2 y}{dx^2}$.

$y' + 5y$ can be written as $Dy + 5y = (D+5)y$

We can operate on this using D:

$$D(D+5)y = D(Dy+5y) = D^2 y + 5Dy$$

$$= (D^2 + 5D)y$$

Further, we can operate on $(D+5)y$ using $D+2$:

$$D(D+5)y + 2(D+5)y$$

$$(D^2+5D)y + (2D+10)y$$

$$(D^2 + 7D + 10)y = (D+2)(D+5)y$$

$$y' + 5y = e^{3x} \Rightarrow Dy + 5y = e^{3x}$$

$$(D+2)(D+5)y = (D+2)e^{3x} = \frac{d}{dx}(e^{3x}) + 2e^{3x}$$

$$= 3e^{3x} + 2e^{3x} = 5e^{3x}$$

4.9 - Solving Systems of Linear DEs by Elimination

The DE $y' + 2y + 3x = -1$ can be written in terms of the differential operator D, where D represents a derivative. In this form, the DE becomes $Dy + 2y + 3x = -1$. This can also be written $(D+2)y + 3x = -1$. Our work in this lesson makes use of this notation. For this lesson, x and y are functions of t, so $Dx = \frac{dx}{dt}$, $Dy = \frac{dy}{dt}$, and $D^2x = \frac{d^2x}{dt^2}$, etc.

Solve the given system of differential equations by elimination.

Example:

$$\begin{aligned} (D+1)x + (D-1)y &= 2 \quad (1)\\ 3x + (D+2)y &= -1 \quad (2)\end{aligned}$$

$3\cdot(1) - (D+1)(2)$

$3(D+1)x + 3(D-1)y = 6$

$-3(D+1)x - (D+1)(D+2)y = (D+1)1$

$[3D-3-(D^2+3D+2)]y = 6+0+1$

$(-D^2-5)y = 7 \Rightarrow (D^2+5)y = -7$

$y'' + 5y = -7$ $\quad y_c(t) = c_1\cos\sqrt{5}t + c_2\sin\sqrt{5}t$

$y_p = A \Rightarrow A = -\frac{7}{5}$

$y(t) = c_1\cos\sqrt{5}t + c_2\sin\sqrt{5}t - \frac{7}{5}$

(2) $3x + (D+2)y = -1 \Rightarrow x = -\frac{1}{3} - \frac{1}{3}(D+2)y$

$-\frac{1}{3}Dy = \frac{\sqrt{5}}{3}c_1\sin\sqrt{5}t - \frac{\sqrt{5}}{3}c_2\cos\sqrt{5}t$

$-\frac{2}{3}y = -\frac{2}{3}c_1\cos\sqrt{5}t - \frac{2}{3}c_2\sin\sqrt{5}t + \frac{14}{15}$

$$x(t) = \left(-\frac{2}{3}c_1 - \frac{\sqrt{5}}{3}c_2\right)\cos\sqrt{5}t + \left(\frac{\sqrt{5}}{3}c_1 - \frac{2}{3}c_2\right)\sin\sqrt{5}t + \frac{3}{5}$$

$$y(t) = c_1\cos\sqrt{5}t + c_2\sin\sqrt{5}t - \frac{7}{5}$$

↑ for presentation

Alternative approach for finding x (Assuming we already found y):

ple:
$$\begin{aligned} (D+1)x + (D-1)y &= 2 \quad (1)\\ 3x + (D+2)y &= -1 \quad (2)\end{aligned}$$

Now elim. y

$(D+2)(1) - (D-1)(2)$

$$(D+2)(D+1)x + (D+2)(D-1)y = (D+2)2 = 4$$
$$-3(D-1)x - (D-1)(D+2)y = (D-1)1 = -1$$
$$(D^2+3D+2-3D+3)x = 3$$
$$(D^2+5)x = 3 \Rightarrow m = \pm\sqrt{5}i$$

$$x_c(t) = c_3\cos\sqrt{5}t + c_4\sin\sqrt{5}t$$
$$x_p = A \Rightarrow A = \frac{3}{5}$$
$$x(t) = c_3\cos\sqrt{5}t + c_4\sin\sqrt{5}t + \frac{3}{5}$$
$$y(t) = c_1\cos\sqrt{5}t + c_2\sin\sqrt{5}t - \frac{7}{5}$$

We have too many constants. To reconcile them: $x(t)$, $y(t) \Rightarrow$ (2) ← arbitrary choice

ple:
$$\begin{aligned} (D+1)x + (D-1)y &= 2 \quad (1)\\ 3x + (D+2)y &= -1 \quad (2)\end{aligned}$$

$3C_3\cos\sqrt{5}t + 3C_4\sin\sqrt{5}t + \frac{9}{5} - C_1\sqrt{5}\sin\sqrt{5}t$

$+ C_2\sqrt{5}\cos\sqrt{5}t + 2C_1\cos\sqrt{5}t + 2C_2\sin\sqrt{5}t - \frac{14}{5} = -1$

cosine terms: $3C_3 + C_2\sqrt{5} + 2C_1 = 0$ $\quad C_3 = -\frac{2}{3}C_1 - \frac{\sqrt{5}}{3}C_2$

sine terms: $3C_4 - C_1\sqrt{5} + 2C_2 = 0$ $\quad C_4 = \frac{\sqrt{5}}{3}C_1 - \frac{2}{3}C_2$,

const: $\frac{9}{5} - \frac{14}{5} = -1$ ✓

as before.

Example:

$$\frac{dx}{dt} + \frac{dy}{dt} = e^t$$

$$-\frac{d^2x}{dt^2} + \frac{dx}{dt} + x + y = 0$$

$Dx + Dy = e^t$

$-D^2x + Dx + x + y = 0$

$(-D^2 + D + 1)x$

① $- D$ ②

$D^3x - D^2x = e^t \Rightarrow m^3 - m^2 = 0$

$m^2(m-1) = 0 \Rightarrow m = 0$ (repeated), 1

$x_c(t) = C_1 + C_2t + C_3e^t$

$x_p = Ate^t$ (Avoiding redundancy)

$x_p' = (A + At)e^t$, $x_p'' = (2A + At)e^t$, $x_p''' = (3A + At)e^t$

omitting e^t: $3A + \cancel{At} - (2A + \cancel{At}) = 1 \Rightarrow A = 1$

$x_p = te^t \Rightarrow x(t) = C_1 + C_2t + C_3e^t + te^t$

$x(t) \to$ ② $\quad Dx = C_2 + C_3e^t + e^t + te^t$

↳ $-D^2x + Dx + x + y = 0 \Rightarrow y = D^2x - Dx - x$

$$D^2x = c_3e^t + 2e^t + te^t$$

$$y = c_3e^t + 2e^t + te^t - c_2 - c_3e^t - e^t - te^t$$
$$- c_1 - c_2t - c_3e^t - te^t$$

$$\boxed{\begin{aligned} x(t) &= c_1 + c_2t + c_3e^t + te^t \\ y(t) &= -c_1 - c_2 - c_2t - c_3e^t - te^t + e^t \end{aligned}}$$

Example:

$$\begin{array}{lrrrl} (1) & Dx & & +z & = e^t \\ (2) & (D-1)x & +Dy & +Dz & = 0 \\ (3) & x & +2y & +Dz & = e^t \end{array}$$

Elim y: $2(2) - D(3)$ $\quad (2D-2)x + 2Dy + 2Dz = 0$

$$-Dx - 2Dy - D^2z = -e^t$$

(4) $(D-2)x + (-D^2+2D)z = -e^t$

(1) $Dx + z = e^t$

Elim z: $-(4) + (-D^2+2D)(1)$ $\quad (-D+2)x + (D^2-2D)z = e^t$

$(-D^2+2D)e^t = -(e^t)'' + 2(e^t)'$ $\quad (-D^3+2D^2)x + (-D^2+2D)z = e^t$

$$(-D^3 + 2D^2 - D + 2)x = 2e^t$$

$$m^2(m-2) + (m-2) = 0 \Rightarrow m = 2, \pm i$$

$$x_c = c_1e^{2t} + c_2\cos t + c_3\sin t$$

$$x_p = Ae^t = x_p' = x_p'' = x_p'''$$

$$-A + 2A - A + 2A = 2 \Rightarrow A = 1$$

$$x(t) = c_1e^{2t} + c_2\cos t + c_3\sin t + e^t$$

① $z = e^t - Dx = e^t - (2c_1e^{2t} - c_2\sin t + c_3\cos t + e^t)$

$z(t) = -2c_1e^{2t} + c_2\sin t - c_3\cos t$

using ③ $y(t) = \frac{3}{2}c_1e^{2t} - c_2\cos t - c_3\sin t$

$$x(t) = c_1e^{2t} + c_2\cos t + c_3\sin t + e^t$$
$$y(t) = \frac{3}{2}c_1e^{2t} - c_2\cos t - c_3\sin t$$
$$z(t) = -2c_1e^{2t} + c_2\sin t - c_3\cos t$$

Example:

$$\frac{dx}{dt} = -x + z \qquad ① \quad (D+1)x - z = 0$$
$$\frac{dy}{dt} = -y + z \qquad ② \quad (D+1)y - z = 0$$
$$\frac{dz}{dt} = -x + y \qquad ③ \quad x - y + Dz = 0$$

Elim y: ② + $(D+1)$③ $\quad (D+1)y - z = 0$

$$(D+1)x - (D+1)y + (D^2+D)z = 0$$

④ $(D+1)x + (D^2+D-1)z = 0$

$-$① $-(D+1)x + z = 0$

$(D^2+D)z = 0 \Rightarrow m^2 + m = 0$

$m = 0, -1 \Rightarrow z(t) = c_1 + c_2e^{-t}$

① ④ elim z ④ $(D+1)x + (D^2+D-1)z = 0$

(D^2+D-1) ① $(D^2+D-1)(D+1)x - (D^2+D-1)z = 0$

$[(D^2+D-1)(D+1) + (D+1)]x = 0$

$(D+1)(D^2+D-1+1)x = 0 \Rightarrow D(D+1)^2 x = 0$

$m = 0, -1$ repeated

$x(t) = c_3 + c_4 e^{-t} + c_5 t e^{-t}$

③: $y = x + Dz = c_3 + c_4 e^{-t} + c_5 t e^{-t} - c_2 e^{-t}$

$y(t) = c_3 + (c_4 - c_2) e^{-t} + c_5 t e^{-t}$

To reconcile the constants:

① $(D+1)x = z$

$-c_4 e^{-t} + c_5(1-t)e^{-t} + c_3 + c_4 e^{-t} + c_5 t e^{-t}$

Dx $\qquad$ x

$= c_1 + c_2 e^{-t}$

z

e^{-t}: $-c_4 + c_5 + c_4 = c_2 \Rightarrow c_2 = c_5$

const: $c_3 = c_1$

$x(t) = c_3 + c_4 e^{-t} + c_5 t e^{-t}$

$y(t) = c_3 + (c_4 - c_5) e^{-t} + c_5 t e^{-t}$

$z(t) = c_3 + c_5 e^{-t}$