4.7 - Cauchy-Euler Equations

Definition: A **Cauchy-Euler equation** is a differential equation of the form $a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$. Earlier in the chapter we found solutions of the form $y = e^{mx}$. Here we will find solutions of the form $y = x^m$.

Solve the given differential equation.

Example: $x^2 y'' + 3xy' - 4y = 0$

$y' = m x^{m-1} \Rightarrow y'' = m(m-1) x^{m-2}$

$x^2 m(m-1) x^{m-2} + 3x\, m x^{m-1} - 4 x^m = 0$

$m(m-1) x^m + 3m x^m - 4 x^m = 0$

$m(m-1) + 3m - 4 = 0$

$m^2 + 2m - 4 = 0$

$m^2 + 2m + 1 = 4 + 1$

$(m+1)^2 = 5 \Rightarrow m = -1 \pm \sqrt{5}$

$$y = c_1 x^{-1+\sqrt{5}} + c_2 x^{-1-\sqrt{5}}$$

For $ax^2 y'' + bxy' + cy = 0$, the auxiliary equation is

$am(m-1) + bm + c = 0$

$am^2 - am + bm + c = 0$

One of three results can stem from this quadratic equation:

Case I: $m_1 \neq m_2 \in \mathbb{R}$

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

Case II: $m_1 = m_2 = m$ (repeated)

$y_1 = x^m \Rightarrow y_2 = x^m \int \frac{e^{-\frac{b}{a}\int \frac{1}{x}dx}}{x^{(a-b)/a}} dx = x^m \int \frac{x^{-b/a}}{x^{1-b/a}} dx$

$ax^2y'' + bxy' + cy = 0 \Rightarrow y'' + \frac{b}{ax}y' + \frac{c}{ax^2}y = 0$

$\rightarrow (-a+b)^2 - 4ac = 0 \Rightarrow \frac{a-b}{2a} = m \Rightarrow y_1 = x^{\frac{a-b}{2a}}$

... $y_2 = x^m \ln x$

$$y_c = c_1 x^m + c_2 x^m \ln x$$

Case III: $m_1 = \alpha + \beta i$, $m_2 = \alpha - \beta i$

$$y = x^\alpha [c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]$$

(This is again due to Euler's formula and the fact that $x^\alpha = e^{\alpha \ln x}$)

Nonhomogeneous:

$ax^2y'' + bxy' + cy = g(x)$

↳ Std form → $y'' + P(x)y' + Q(x)y = f(x)$

↳ variation of parameters

Example: $4x^2y'' + y = 0$

$y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

$4m(m-1) + 1 = 0 \Rightarrow 4m^2 - 4m + 1 = 0$

$(2m-1)^2 = 0 \Rightarrow m = \frac{1}{2}$, repeated

$$y = c_1 x^{1/2} + c_2 x^{1/2} \ln x$$

Example: $x^2y'' - 7xy' + 41y = 0$

$m(m-1) - 7m + 41 = 0 \Rightarrow m^2 - 8m + 41 = 0$

$m^2 - 8m + 16 = -41 + 16$

$(m-4)^2 = -25 \Rightarrow m = 4 \pm 5i$

$$y = x^4 \left[c_1 \cos(5 \ln x) + c_2 \sin(5 \ln x) \right]$$

A Cauchy-Euler equation can be converted to an equation with constant coefficients using the substitution $x = e^t$ and the Chain Rule.

Rationale: The general solution to $ay'' + by' + cy = 0$, a, b, c constants, is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \qquad (*)$$

the general solution to $ax^2y'' + bxy' + cy = 0$ (a, b, c constants) is $y = c_1 x^{m_1} + c_2 x^{m_2}$. But

$$x = e^{\ln x} \Rightarrow x^{m_1} = e^{m_1 \ln x}$$

$$\Rightarrow y = c_1 e^{m_1 \ln x} + c_2 e^{m_2 \ln x}.$$ Let $x = e^t$. Then $y = c_1 e^{m_1 t} + c_2 e^{m_2 t}$, as in $(*)$.

Example: Use the substitution $x = e^t$ to transform the given Cauchy-Euler equation to a differential equation with constant coefficients. Solve the original equation by solving the new equation using techniques previously learned.

$x^2y'' - 9xy' + 25y = 0$

$y = x^m$; $x = e^t \Rightarrow t = \ln x$

$y(x)$ becomes $y(t(x))$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\left(\frac{d^2y}{dt^2}\frac{1}{x}\right) = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$\frac{dy}{dt}$ | t | $\frac{dt}{dx}$ | x

$$x^2 \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - 9x \frac{1}{x}\frac{dy}{dt} + 25y = 0$$

$$\frac{d^2y}{dt^2} - 10\frac{dy}{dt} + 25y = 0$$

$$m^2 - 10m + 25 = 0$$

$$m = 5, \text{ repeated.}$$

$$y = c_1 e^{5t} + c_2 t e^{5t}$$

$$y = c_1 e^{5\ln x} + c_2 \ln x\, e^{5\ln x}$$

$$\boxed{y = c_1 x^5 + c_2 x^5 \ln x}$$

Homog → auxiliary eqn.

↑ @

↳ Nonhomog.

↙ ↘

g(x) has
linear
poly
exp
sin / cos
+ , ×

↓

$y_p = Ax^2 + Bx + C$

(Avoid redundancy w/ y_c)

g(x) has anything else

Variation of Parameters

Std form $g(x) \to f(x)$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

C_1 of W_1 : $\begin{matrix} 0 \\ f(x) \end{matrix}$

or : $\begin{matrix} 0 \\ 0 \\ f(x) \end{matrix}$

$\begin{matrix} 0 \\ 0 \\ 0 \\ f(x) \end{matrix}$

If Aux eqn → $m = 3$, repeated mult. 2

$y_1 = e^{3x}$, $y_2 = xe^{3x}$

Reduction of order: $y_2 = u y_1$

$w = u'$

Process

formula: $y_2 = y_1 \int \frac{e^{-\int P dx}}{y_1^2} dx$