9.2 - Runge-Kutta Methods

A generalization of Euler's formula is $y_{n+1} = y_n + h(w_1k_1 + w_2k_2 + \cdots + w_mk_m)$, where the quantity in parentheses is a weighted average of slopes.

Notes: $\sum_{i=1}^{m} w_i = 1$, k_i are recursively defined, and m is the **order** of the method.

Definition: For $y_{n+1} = y_n + hf(x_n, y_n)$, we have $m = 1$, $w_1 = 1$, and $k_1 = f(x_n, y_n)$. Thus Euler's method is a **first-order Runge-Kutta method**.

For reference: A second-order Runge-Kutta method has the form $y_{n+1} = y_n + h(w_1k_1 + w_2k_2)$, where $k_1 = f(x_n, y_n)$ and $k_2 = f(x_n + \alpha h, y_n + \beta hk_1)$. When $w_1 + w_2 = 1$, $w_2\alpha = 1/2$, and $w_2\beta = 1/2$, the results agree with a Taylor polynomial of degree 2. (Note that this is only one choice for the constants; there are others, though we won't be exploring that.) If we choose $w_2 = 1/2$, then $w_1 = 1/2$, and $\alpha = \beta = 1$. The result agrees with the improved Euler's method, thus making it a **second-order Runge-Kutta method**.

A fourth-order Runge-Kutta method (RK4)

An RK4 method uses a weighted average of four slopes to approximate y-values. The basis of the technique is to find parameters so the formula

$y_{n+1} = y_n + h(w_1k_1 + w_2k_2 + w_3k_3 + w_4k_4)$, where

$$k_1 = f(x_n, y_n)$$
$$k_2 = f(x_n + \alpha_1 h, y_n + \beta_1 hk_1)$$
$$k_3 = f(x_n + \alpha_2 h, y_n + \beta_2 hk_1 + \beta_3 hk_2)$$
$$k_4 = f(x_n + \alpha_3 h, y_n + \beta_4 hk_1 + \beta_5 hk_2 + \beta_6 hk_3)$$

agrees with a 4th-degree Taylor polynomial. The resulting system has 11 equations in 13 unknowns, and common choices for the parameters yields

$$
\begin{aligned}
y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
k_1 &= f(x_n, y_n) \\
k_2 &= f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1) \\
k_3 &= f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2) \\
k_4 &= f(x_n + h, y_n + hk_3)
\end{aligned}
$$

Local and global truncation errors using RK4 are $O(h^5)$ and $O(h^4)$, respectively.

Example: Use the RK4 method with $h = 0.1$ to obtain a four-decimal approximation of the indicated value.

$y' = 4x - 2y, \qquad y(0) = 2; \qquad y(0.5)$