7.3 - Operational Properties I

7.3a

Theorem: First Translation Theorem

If $\mathscr{L}\{f(t)\} = F(s)$ and a is any real number, then $\mathscr{L}\{e^{at}f(t)\} = F(s-a)$.

Example: Find either $F(s)$ or $f(t)$, as indicated.

$\mathscr{L}\left\{t^{10}e^{-7t}\right\}$

$\mathscr{L}\left\{e^{-2t}\cos 4t\right\}$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\}$$

$$\mathscr{L}^{-1}\left\{\frac{(s+1)^2}{(s+2)^4}\right\}$$

7.3b

Example: Use the Laplace transform to solve the initial-value problem.

$y' - y = 1 + te^t, \qquad y(0) = 0$

$2y'' + 20y' + 51y = 0, \qquad y(0) = 2, \quad y'(0) = 0$

Example: Solve the boundary-value problem.

$$y'' + 8y' + 20y = 0, \qquad y(0) = 0, \quad y'(\pi) = 0$$

7.3c

The First Translation Theorem showed how to translate on the s-axis. A translation on the t-axis is essentially an adjustment for moving the lower limit in the definition of a Laplace transform. This is accomplished via the **Heaviside function** or **unit step function.**

Definition: The **unit step function** is

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

Theorem: **Second Translation Theorem**

If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then $\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s)$.

If we let $f(t) = 1$, then we have $\mathscr{L}\{\mathscr{U}(t-a)\} = \dfrac{e^{-as}}{s}$.

Example: Find either $F(s)$ or $f(t)$, as indicated.

$\mathscr{L}\{(3t+1)\mathscr{U}(t-1)\}$

Alternative form of the second translation theorem

$\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}$

7.3d

Inverse form of the Second Translation Theorem

If $f(t) = \mathscr{L}^{-1}\{F(s)\}$, then $\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$.

<u>**Example:**</u> Find either $F(s)$ or $f(t)$, as indicated.

$$\mathscr{L}^{-1}\left\{\frac{(1+e^{-2s})^2}{s+2}\right\}$$

$$\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s^2(s-1)}\right\}$$

Example: Use the Laplace transform to solve the given initial-value problem.

$y' + y = f(t), \qquad y(0) = 0$, where

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ -1 & t \geq 1 \end{cases}$$