

6.4 - Special Functions

6.4a

Bessel's equation of order ν

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

Example: Find the general solution of the given differential equation on $(0, \infty)$.

$$16x^2 y'' + 16x y' + (16x^2 - 1) y = 0$$

Bessel Functions of the First and Second Kinds

Assuming a solution of the form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ and substituting into Bessel's equation, we have

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} - \sum_{n=0}^{\infty} \nu^2 c_n x^{n+r} = 0$$

$$\implies \sum_{n=0}^{\infty} [(n+r)(n+r) - \nu^2] c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} = 0$$

$$\text{Reindexing yields } \sum_{k=0}^{\infty} [(k+r)^2 - \nu^2] c_k x^{k+r} + \sum_{k=2}^{\infty} c_{k-2} x^{k+r} = 0$$

For $k = 0$ we find $r = \pm \nu$ and $c_1 = 0$.

$$\text{Then } c_k = -\frac{c_{k-2}}{(k+r)^2 - \nu^2}, k = 2, 3, 4, \dots$$

A standard choice for c_0 is $c_0 = \frac{1}{2^\nu \Gamma(1+\nu)}$, where $\Gamma(1+\nu)$ is the gamma function.

Definition: The **gamma function** is the integral-defined function

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$$



The function $y = x^\nu \sum_{n=0}^{\infty} c_{2n} x^{2n}$ is then

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1 + \nu + n)} \left(\frac{x}{2}\right)^{2n+\nu} = J_\nu(x)$$

We also have that $J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1 - \nu + n)} \left(\frac{x}{2}\right)^{2n-\nu}$

6.4b

Definition: The functions $J_\nu(x)$ and $J_{-\nu}(x)$ defined above are **Bessel functions of the first kind** of order ν and $-\nu$, respectively and converge at least on $(0, \infty)$.

If ν is not an integer, then the general solution to Bessel's equation is

$$y = c_1 J_\nu(x) + c_2 J_{-\nu}(x).$$

Definition: The function $Y_\nu(x) = \frac{\cos(\nu\pi)J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}$ (ν a non-integer) is a **Bessel function of the second kind**. It can be shown to be linearly independent with $J_\nu(x)$.

If ν is an integer, then it can be shown that $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly dependent. However, $J_\nu(x)$ and $Y_\nu(x)$ are independent. As such, for an integer value of ν , we use $J_\nu(x)$ and $Y_\nu(x)$ (in this case, $Y_\nu(x)$ is actually defined by a limiting process as $\nu \rightarrow m$, where m is an integer). In this case we can give the general solution to Bessel's equation as $y = c_1 J_\nu(x) + c_2 Y_\nu(x)$.

Example: Find the general solution of the given differential equation on $(0, \infty)$.

$$x^2 y'' + x y' + (x^2 - 1)y = 0$$

Alternate forms of Bessel's equation

Using a substitution we can find that the general solution to the DE $x^2 y'' + x y' + (\alpha^2 x^2 - \nu^2) y = 0$ is $y = c_1 J_\nu(\alpha x) + c_2 Y_\nu(\alpha x)$.

Definition: The preceding differential equation is called the **parametric Bessel equation of order ν** .

Definition: The differential equation $x^2 y'' + x y' - (x^2 + \nu^2) y = 0$ is the **modified Bessel equation of order ν** . It can be converted to a Bessel equation by the substitution $t = ix$, where $i^2 = -1$; the result is...

Definition: The **modified Bessel function of the first kind** of order ν , $I_\nu(x) = i^{-\nu} J_\nu(ix)$.

The general solution to the modified Bessel equation of order ν (ν not an integer) is $y = c_1 I_\nu(x) + c_2 I_{-\nu}(x)$.

Definition: As before, for non-integer values of ν we define the **modified Bessel function of the second kind**, $K_\nu(x) = \frac{\pi I_{-\nu}(x) - I_\nu(x)}{2 \sin \nu \pi}$ and extend this definition to integer values using a limit process similar to that referenced when defining $Y_\nu(x)$.

And as before, the general solution to the modified Bessel equation of order ν (for ν an integer) is $y = c_1 I_\nu(x) + c_2 K_\nu(x)$. A parametric form of the modified Bessel equation and associated solution exists such as we saw above.

Definition: An equation of the form $x^2 y'' + x y' - (\alpha^2 x^2 + \nu^2) y = 0$ is a **parametric form of the modified Bessel equation of order ν** and can be obtained from a modified Bessel equation using a change of variables.

The general solution for such an equation is $y = c_1 I_\nu(\alpha x) + c_2 K_\nu(\alpha x)$.

Example: Find the general solution of the given differential equation on $(0, \infty)$.

$$x^2 y'' + x y' - (2x^2 + 64)y = 0$$

Example: The general solution to $y'' + \frac{1-2a}{x}y' + \left(b^2 c^2 x^{2c-2} + \frac{a^2 - p^2 c^2}{x^2}\right)y = 0$, $p \geq 0$ is $y = x^a [c_1 J_p(b x^c) + c_2 Y_p(b x^c)]$. Use this to find the general solution of the given differential equation on $(0, \infty)$.

$$x y'' + 3y' + x y = 0$$

Legendre polynomials

Definition: The differential equation

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$$

is **Legendre's equation of order n** .

Note that $x = 0$ is an ordinary point. Starting with $y = \sum_{k=0}^{\infty} c_k x^k, \dots$

$$\text{leads to } 2c_2 + 6c_3x - 2c_1x + n(n+1)c_0 + n(n+1)c_1x + \sum_{j=2}^{\infty} \left\{ (j+2)(j+1)c_{j+2} + [-j(j-1) - 2j + n(n+1)] c_j \right\} x^j = 0$$

$$\text{We find that } c_{j+2} = -\frac{(n-j)(n+j+1)}{(j+2)(j+1)} c_j, j = 2, 3, 4, \dots$$

$$c_2 = -\frac{n(n+1)}{2} c_0$$

$$c_4 = -\frac{(n-2)(n+3)}{4 \cdot 3} c_2 = \frac{(n-2)n(n+1)(n+3)}{4!} c_0$$

This terminates if n is even.

$$c_3 = -\frac{(n-1)(n+2)}{6} c_1$$

$$c_5 = -\frac{(n+4)(n-3)}{5 \cdot 4} c_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} c_1$$

This terminates if n is odd.

Traditional values for c_i are

$$\text{For } n = 0, c_0 = 1; \text{ for } n = 2, 4, 6, \dots, c_0 = (-1)^{n/2} \frac{1 \cdot 3 \cdots (n-1)}{2 \cdot 4 \cdots n}$$

$$\text{For } n = 1, c_1 = 1; \text{ for } n = 3, 5, 7, \dots, c_1 = (-1)^{(n-1)/2} \frac{1 \cdot 3 \cdots n}{2 \cdot 4 \cdots (n-1)}$$

Definition: For each value of n the result is a **Legendre polynomial of order n** .

The first six Legendre polynomials are

$$P_0(x) = 1$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_1(x) = x$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$