6.3 - Solutions About Singular Points

6.3a

A singular point for the equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ (*) is a point $x = x_0$ such that $a_2(x_0) = 0$. For our work with solving differential equations, we will only consider $x_0 = 0$. Standard form for (*) is $y'' + P(x)y' + Q(x)y = 0$. Multiplying by x^2 (or by $(x - x_0)^2$ if $x_0 \neq 0$) yields
$x^2y'' + x^2P(x)y' + x^2Q(x)y = 0$. By relabeling $p(x) = xP(x)$, $q(x) = x^2Q(x)$, we get $x^2y'' + xp(x)y' + q(x)y = 0$.

Definition: $x = x_0$ is a **regular singular point** if $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2Q(x)$ are analytic at $x = x_0$ and an **irregular singular point** if either $p(x)$ or $q(x)$ is not analytic at $x = x_0$.

Example: Determine the singular points of the given differential equation. Classify each singular point as regular or irregular.

$(x^2 - 9)^2y'' + (x + 3)y' + 2y = 0$

Theorem: (Frobenius' Theorem) If $x = x_0$ is a regular singular point of $(*)$, then there is at least one solution of the form
$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$, where the number r is a constant to be determined.

6.3b

Example: $x = 0$ is a regular singular point of the given differential equation. Show that the indicial roots of the singularity do not differ by an integer. Use the method of Frobenius to obtain two linearly independent solutions about $x = 0$. Form the general solution on $(0, \infty)$.

$2xy'' + 5y' + xy = 0$

6.3c

The equation $r(2r + 3) = 0$ is an indicial equation and can be generalized as follows:

6.3d

Example: Use the method of Frobenius to obtain two linearly independent solutions about the regular singular point $x = 0$.

$9x^2y'' + 9x^2y' + 2y = 0$

6.3e

If r_1 and r_2 differ by an integer–that is, if $r_2 - r_1 \in \mathbb{Z}$–we can use r_1 to find y_1 and then reduction of order to find r_2.

Example: $x = 0$ is a regular singular point of the given differential equation. Show that the indicial roots of the singularity differ by an integer. Use the method of Frobenius to obtain at least one series solution about $x = 0$. Use reduction of order and a CAS to find a second solution. Form the general solution on $(0, \infty)$.

$$y'' + \frac{3}{x}y' - 2y = 0$$