5.2 - Linear Models: Boundary-Value Problems

5.2a

Consider $y'' + 3y = 0,\ y(0) = 0,\ y(\pi) = 0$

Now consider $y'' + 4y = 0,\ y(0) = 0,\ y(\pi) = 0$

Definition: A number λ for which there exist nontrivial solutions to $y'' + P(x)y' + \lambda Q(x)y = 0,\ y(a) = 0,\ y(b) = 0$ is called an **eigenvalue**, and a solution $y = ay_1 + by_2$ that satisfies the BVP is an **eigenfunction** associated with the eigenvalue λ.

Example: Determine eigenvalues and eigenfunctions for
$y'' + \lambda y = 0,\ y(0) = 0,\ y(L) = 0\ (L > 0)$.

5.2b

Consider a vertical beam of length L with load P applied to one end or a road with compressive force P applied to one end. Theory of elasticity provides the differential equation $EI\dfrac{d^2y}{dt^2} = -Py$ or $EI\dfrac{d^2y}{dt^2} + Py = 0$ which in standard form is

$$\frac{d^2y}{dt^2} + \frac{P}{EI}y = 0$$

The product EI is the flexural rigidity of the beam, E is Young's modulus of elasticity, and I is the moment of inertia of a cross-section of the beam (the shape's inherent resistance to bending).

Considering $P(x) = 0,\ Q(x) = 1$, and letting $\lambda = \dfrac{P}{EI}$, this has the form

$$y'' + \lambda y = 0$$

Considering a beam of length L, the bending moment (response or resistance to load) $M(x)$ is related to the load per unit length $w(x)$ by $\dfrac{d^2M}{dx^2} = w(x)$ (this is from theory of elasticity). Assuming down is positive, $M(x) = EI\kappa$ is proportional to the curvature of the elastic curve. From multivariable calculus, curvature is $\kappa = \dfrac{y''}{[1+(y')^2]^{3/2}}$. For small deflection, $y' = 0$, so $M = EIy''$. We see that $M'' = w(x)$ becomes

$$EI\frac{d^4y}{dx^4} = w(x)$$

Definition: The graph of the function $y(x)$ is the **deflection curve** of the beam.

For the deflection $y(x)$, we have the following:
y' is the slope of the curve;
y'' is the bending moment;
y''' is the shear force.

5.2c

Example: Solve the differential equation $EI\frac{d^4y}{dx^4} = w(x)$ subject to the appropriate boundary conditions. The beam is of length L, and w_o is a constant.

(a) The beam is simply supported at both ends, and
$w(x) = w_0$, $0 < x < L$.

(b) Use a graphing utility to graph the deflection curve when $w_0 = 24EI$ and $L = 1$.