# 8.2 – Compositions and Inverse Transformations

**Definition:** If $T : V \to W$ is a linear transformation from a vector space $V$ to a vector space $W$, then $T$ is said to be **one-to-one** if $T$ maps distinct vectors in $V$ into distinct vectors in $W$.
OR $T : V \to W$ is one-to-one if and only if for each vector $\mathbf{w}$ in the range of $T$, there is exactly one vector $\mathbf{v}$ in $V$ such that $T(\mathbf{v}) = \mathbf{w}$.
OR $T : V \to W$ is one-to-one if and only if $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$.

**Definition:** If $T : V \to W$ is a linear transformation from a vector space $V$ to a vector space $W$, then $T$ is said to be **onto** (or **onto $W$**) if every vector in $W$ is the image of at least one vector in $V$.

**Theorem 8.2.1** If $T : V \to W$ is a linear transformation, then $T$ is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$.

**#4** Determine whether the linear transformation is one-to-one by finding its kernel and then applying Theorem 8.2.1.

a. $T : R^2 \to R^3$, where $T(x,\ y) = (x - y,\ y - x,\ 2x - 2y)$

b. $T : R^2 \to R^2$, where $T(x,\ y) = (0,\ 2x + 3y)$

c. $T : R^2 \to R^2$, where $T(x,\ y) = (x + y,\ x - y)$

**Theorem 8.2.X** If $T : V \to W$ is a one-to-one linear transformation and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a linearly independent subset of $V$, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)\}$ is linearly independent.

**#5** Determine whether multiplication by $A$ is one-to-one by computing the nullity of $A$ and then applying Theorem 8.2.1.

a. $A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ -3 & 6 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 3 & 1 & 7 \\ 2 & 7 & 2 & 4 \\ -1 & -3 & 0 & 0 \end{bmatrix}$

**Theorem 8.2.2** If $V$ and $W$ are finite-dimensional vector spaces with the same dimension, and if $T : V \to W$ is a linear transformation, then the following statements are equivalent.

a) $T$ is one-to-one.

b) $\ker(T) = \{\mathbf{0}\}$.

c) $T$ is onto [i.e., $R(T) = W$].

**Theorem 8.2.3** If $T_A : R^n \to R^m$ is a matrix transformation, then

a) $T_A$ is one-to-one if and only if the columns of $A$ are linearly independent.

b) $T_A$ is onto if and only if the columns of $A$ span $R^m$.

**#14** Use Theorem 8.2.3 to determine whether multiplication by $A$ is one-to-one, onto, both, or neither. Justify your answer.

a. $A = \begin{bmatrix} 9 & -3 \\ -4 & 2 \\ 1 & 1 \end{bmatrix}$

b. $A = \begin{bmatrix} 3 & -3 & 1 & 1 \\ 6 & -6 & 0 & 2 \\ 9 & -9 & 1 & 3 \end{bmatrix}$

c. $A = \begin{bmatrix} 3 & -9 \\ -1 & 3 \end{bmatrix}$

d. $A = \begin{bmatrix} 2 & 3 & 8 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

**Theorem 8.2.4** Equivalent Statements (extends Theorems 4.9.8, 5.1.5 and 6.4.5, the last two of which we have not seen)
If $A$ is an $n \times n$ matrix, then the following statements are equivalent.
a) $A$ is invertible.
b)$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
c) The reduced row echelon form of $A$ is $I_n$.
d) $A$ is expressible as a product of elementary matrices.
e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
g) $\det(A) \neq 0$.
h) The column vectors of $A$ are distinct and linearly independent.
i) The row vectors of $A$ are distinct and linearly independent.
j) The column vectors of $A$ span $R^n$.
k) The row vectors of $A$ span $R^n$.
l) The column vectors of $A$ form a basis for $R^n$.
m) The row vectors of $A$ form a basis for $R^n$.
n) $A$ has rank $n$.
o) $A$ has nullity 0.
p) The orthogonal complement of the null space of $A$ is $R^n$.
q) The orthogonal complement of the row space of $A$ is $\{\mathbf{0}\}$.
r) $\lambda = 0$ is not an eigenvalue of $A$. [This is from Theorem 5.1.5.]
s) $A^T A$ is invertible. [This is from Theorem 6.4.5.]
t) The kernel of $T_A$ is $\{\mathbf{0}\}$.
u) The range of $T_A$ is $R^n$.
v) $T_A$ is one-to-one.

**Definition:** If $T : V \to W$ is a one-to-one linear transformation with range $R(T)$, and if $\mathbf{w}$ is any vector in $R(T)$, then there is exactly one vector $\mathbf{v}$ in $V$ for which $T(\mathbf{v}) = \mathbf{w}$. The **inverse of T** maps $\mathbf{w}$ back into $\mathbf{v}$ and is denoted by $T^{-1}$. (This is analogous to **inverse** in Section 1.9.)

**Definition:** (analogous to **composition** in Section 1.9)
If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, then the **composition** of $T_2$ with $T_1$, denoted by $T_2 \circ T_1$ (which is read "$T_2$ circle $T_1$") is the function defined by the formula $(T_2 \circ T_1)(\mathbf{u}) = T_2(T_1(\mathbf{u}))$ where $\mathbf{u}$ is a vector in $U$.

**Theorem 8.2.5** If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, then $(T_2 \circ T_1) : U \to W$ is also a linear transformation.

**Theorem 8.2.6** If $T_1 : U \to V$ and $T_2 : V \to W$ are one-to-one linear transformations, then:
a) $T_2 \circ T_1$ is one-to-one.
b) $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$.

**#38** Let $D(\mathbf{f}) = f'(x)$ and $J(\mathbf{f}) = \int_0^x f(t)dt$ be the derivative and integral linear transformations. Find $(J \circ D)(\mathbf{f})$ for

a. $\mathbf{f}(x) = x^2 + 3x + 2$

b. $\mathbf{f}(x) = \sin x$

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**Key concepts from 8.3:**

Definition 1: A linear transformation $T : V \to W$ that is both one-to-one and onto is said to be an **isomorphism**, and $W$ is said to be **isomorphic** to $V$.

**Theorem 8.3.1** Every real $n$-dimensional vector space is isomorphic to $R^n$. [In this sense, every $n$-dimensional vector space is algebraically equivalent to $R^n$.]

**Theorem 8.3.2** If $S$ is an ordered basis for a vector space $V$, then the coordinate map $\mathbf{u} \overset{T}{\to} (\mathbf{u})_S$ is an isomorphism between $V$ and $R^n$.

**Theorem 8.3.X** A linear transformation $T : V \to W$ is an isomorphism if and only if whenever $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis for $V$, $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)\}$ is a basis for $W$.

**Examples of Natural isomorphisms**

Between $P_{n-1}$ and $R^n$: $a_0 + a_1x + \ldots + a_{n-1}x^{n-1} \overset{T}{\to} (a_0, a_1, \ldots, a_{n-1})$

Between $M_{22}$ and $R^4$: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \overset{T}{\to} (a,\ b,\ c,\ d)$