# 5.1 – Eigenvalues and Eigenvectors

Consider the equation $\begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}$.

**Definition:** If $A$ is an $n \times n$ matrix, then a nonzero vector $\mathbf{x}$ in $R^n$ is called an **eigenvector** of $A$ (or of the matrix operator $T_A$) if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$; that is, $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar $\lambda$. The scalar $\lambda$ is called an **eigenvalue** of $A$ (or of $T_A$), and $\mathbf{x}$ is said to be an **eigenvector corresponding to $\boldsymbol{\lambda}$**.

**Theorem 5.1.1** If $A$ is an $n \times n$ matrix, then $\lambda$ is an eigenvalue of $A$ if and only if it satisfies the equation $\det(\lambda I - A) = 0$.

If $A$ is an $n \times n$ matrix, then the **characteristic equation** of $A$ is $\det(\lambda I - A) = 0$.

**Definition:** The **characteristic polynomial** of an $n \times n$ matrix $A$ results from expanding the characteristic equation of $A$ and has the form
$p(\lambda) = \lambda^n + c_1\lambda^{n-1} + \ldots + c_n$.

The solution space of the system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ is called the **eigenspace** of $A$ corresponding to $\lambda$.

**#6** In each part, find the characteristic equation, the eigenvalues, and bases for the eigenspaces of the matrix.

a. $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ b. $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$ c. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$

**Theorem 5.1.3** If $A$ is an $n \times n$ matrix, the following statements are equivalent.
a) $\lambda$ is an eigenvalue of $A$.
b) $\lambda$ is a solution of the characteristic equation $\det(\lambda I - A) = 0$.
c) The system of equations $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has nontrivial solutions.
d) There is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.

**#9** find the characteristic equation, the eigenvalues, and bases for the eigenspaces of the matrix.

$$\begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$$

**#19** In each part, find the eigenvalues and the corresponding eigenspaces of the stated matrix operator on $R^2$. Use geometric reasoning to find the answers. No computations are needed.

a. Reflection about the line $y = x$.

b. Orthogonal projection onto the $x$-axis.

c. Rotation about the origin through a positive angle of $90^\circ$.

d. Contraction with factor $k$ $(0 \leq k < 1)$

e. Shear in the $x$-direction by a factor $k$ $(k \neq 0)$

**Theorem 5.1.2** If $A$ is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of $A$ are the entries on the main diagonal of $A$.

**#14** Find the characteristic equation of the matrix by inspection.

$$\begin{bmatrix} 9 & -8 & 6 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

**#15** Find the eigenvalues and a basis for each eigenspace of the linear operator defined by the stated formula.
$T(x,\ y) = (x + 4y,\ 2x + 3y)$

**Theorem 5.1.4** A square matrix $A$ is invertible if and only if $\lambda = 0$ is not an eigenvalue of $A$.

**Theorem 5.1.5** Equivalent Statements (extends Theorem 4.9.8)
If $A$ is an $n \times n$ matrix, then the following statements are equivalent.
a)$A$ is invertible.
b)$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
c) The reduced row echelon form of $A$ is $I_n$.
d) $A$ is expressible as a product of elementary matrices.
e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
g) $\det(A) \neq 0$.
h) The column vectors of $A$ are distinct and linearly independent.
i) The row vectors of $A$ are distinct and linearly independent.
j) The column vectors of $A$ span $R^n$.
k) The row vectors of $A$ span $R^n$.
l) The column vectors of $A$ form a basis for $R^n$.
m) The row vectors of $A$ form a basis for $R^n$.
n) $A$ has rank $n$.
o) $A$ has nullity 0.
p) The orthogonal complement of the null space of $A$ is $R^n$.
q) The orthogonal complement of the row space of $A$ is $\{\mathbf{0}\}$.
r) $\lambda = 0$ is not an eigenvalue of $A$.