4.9 – Rank, Nullity, and the Fundamental Matrix Spaces

Theorem 4.9.1 The row space and column space of a matrix A have the same dimension.

Definition: The common dimension of the row space and column space of a matrix A is called the **rank** of A and is denoted by $\text{rank}(A)$; the dimension of the null space of A is called the **nullity** of A and is denoted by $\text{nullity}(A)$. That is, $\text{rank}(A) = \dim[\text{row}(A)] = \dim[\text{col}(A)]$ and $\text{nullity}(A) = \dim[\text{null}(A)]$.

#1 Find the rank and nullity of the matrix A by reducing it to row echelon form.

a. $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -3 & 3 \\ 4 & 8 & -4 & 4 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & 4 \end{bmatrix}$

Theorem 4.9.2 Dimension Theorem for Matrices
If A is a matrix with n columns, then $\text{rank}(A) + \text{nullity}(A) = n$.

Theorem 4.9.3 If A is an $m \times n$ matrix, then
a) $\text{rank}(A)$ = the number of leading variables in the general solution of $A\mathbf{x} = \mathbf{0}$.
b) $\text{nullity}(A)$ = the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$.

Theorem 4.9.4 If $A\mathbf{x} = \mathbf{b}$ is a consistent linear system of m equations in n unknowns, and if A has rank r, then the general solution of the system contains $n - r$ parameters.

The four **fundamental spaces** of a matrix A are row (A), col (A), null (A), and null $\left(A^T\right)$. The null space of A^T is also called the **left null space of A**. The **left nullity of A** is nullity $\left(A^T\right) = \dim\left[\text{null } \left(A^T\right)\right]$.

Theorem 4.9.5 If A is any matrix, then $\text{rank}(A) = \text{rank}\left(A^T\right)$.

#11 Find the dimensions and bases for the four fundamental spaces of the matrix.

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 3 \\ -9 & 0 \end{bmatrix}$$

Definition: If W is a subspace of R^n, then the set of all vectors in R^n that are orthogonal to every vector in W is called the **orthogonal complement** of W and is denoted by the symbol $W^{\perp}$ (pronounced "W perp").

Theorem 4.9.6 If W is a subspace of R^n, then:
a) $W^{\perp}$ is a subspace of R^n.
b) The only vector common to W and $W^{\perp}$ is $\mathbf{0}$, that is, $W \cap W^{\perp} = \{\mathbf{0}\}$.
c) The orthogonal complement of $W^{\perp}$ is W, that is, $\left(W^{\perp}\right)^{\perp} = W$.

Theorem 4.9.7 If A is an $m \times n$ matrix, then:
a) The null space of A and the row space of A are orthogonal complements in R^n.
b) The null space of A^T and the column space of A are orthogonal complements in R^m.

#15 Confirm the orthogonality statements in the two parts of Theorem 4.9.7 for the matrix $A = \begin{bmatrix} 1 & 4 \\ 0 & 3 \\ -9 & 0 \end{bmatrix}$.

#27 Suppose that A is a 3×3 matrix whose null space is a line through the origin in 3-space. Can the row or column space of A also be a line through the origin? Explain.

Theorem 4.9.8 Equivalent Statements (extends Theorem 2.3.8)
If A is an $n \times n$ matrix, then the following statements are equivalent.
a) A is invertible.
b)$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
c) The reduced row echelon form of A is I_n.
d) A is expressible as a product of elementary matrices.
e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
g) $\det(A) \neq 0$.
h) The column vectors of A are distinct and linearly independent.
i) The row vectors of A are distinct and linearly independent.
j) The column vectors of A span R^n.
k) The row vectors of A span R^n.
l) The column vectors of A form a basis for R^n.
m) The row vectors of A form a basis for R^n.
n) A has rank n.
o) A has nullity 0.
p) The orthogonal complement of the null space of A is R^n.
q) The orthogonal complement of the row space of A is $\{\mathbf{0}\}$.