# 4.6 – Dimension

**Theorem 4.6.2** Let $V$ be a finite-dimensional vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any basis for $V$.
a) If a set in $V$ has more than $n$ vectors, then it is linearly dependent.
b) If a set in $V$ has fewer than $n$ vectors, then it does not span $V$.

**Theorem 4.6.1** All bases for a finite-dimensional vector space have the same number of vectors.

**Definition:** The **dimension** of a finite-dimensional vector space $V$ is denoted by $\dim(V)$ and is defined to be the number of vectors in a basis for $V$ (in some physical contexts, "dimension" is referred to as **degrees of freedom**). In addition, the zero vector space is defined to have dimension zero.

Find a basis for the solution space of each homogeneous linear system, and find the dimension of that space.

**#1**
$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ -2x_1 - x_2 + 2x_3 &= 0 \\ -x_1 \qquad + x_3 &= 0 \end{aligned}$$

**#5**

$$\begin{aligned} 2x_1 - 6x_2 + 2x_3 &= 0 \\ 3x_1 - 9x_2 + 3x_3 &= 0 \end{aligned}$$

**#8** In each part, find a basis for the given subspace of $R^4$, and state its dimension.

a. All vectors of the form $(a,\ b,\ c,\ 0)$.

b. All vectors of the form $(a,\ b,\ c,\ d)$, where $d = a + b$ and $c = a - b$.

c. All vectors of the form $(a,\ b,\ c,\ d)$, where $a = b = c = d$.

**Theorem 4.6.4** Let $V$ be an $n$-dimensional vector space, and let $S$ be a set in $V$ with exactly $n$ vectors. Then $S$ is a basis for $V$ if and only if $S$ spans $V$ or $S$ is linearly independent.

**Theorem 4.6.5** Let $S$ be a finite set of vectors in a finite-dimensional vector space $V$.

a) If $S$ spans $V$ but is not a basis for $V$, then $S$ can be reduced to a basis for $V$ by removing appropriate vectors from $S$.

b) If $S$ is a linearly independent set that is not already a basis for $V$, then $S$ can be enlarged to a basis for $V$ by inserting appropriate vectors into $S$.

**#12** Find standard basis vectors for $R^3$ that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for $R^3$.

a. $\mathbf{v}_1 = (-1, 2, 3)$, $\mathbf{v}_2 = (1, -2, -2)$

b. $\mathbf{v}_1 = (1, -1, 0)$, $\mathbf{v}_2 = (3, 1, -2)$

**#18** Find a basis for the subspace of $R^4$ that is spanned by the vectors $\mathbf{v}_1 = (1, 1, 1, 1)$, $\mathbf{v}_2 = (2, 2, 2, 0)$, $\mathbf{v}_3 = (0, 0, 0, 3)$, $\mathbf{v}_4 = (3, 3, 3, 4)$.

**#20** In each part, let $T_A$ be multiplication by $A$ and find the dimension of the subspace of $R^4$ consisting of all vectors $\mathbf{x}$ for which $T_A(\mathbf{x}) = \mathbf{0}$.

a. $\begin{bmatrix} 1 & 0 & 2 & -1 \\ -1 & 4 & 0 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$