4.1 – Real Vector Spaces

Definition: (Vector space)
Let V be an arbitrary nonempty set of objects for which two operations are defined: addition and multiplication by numbers called **scalars**. By **addition** we mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in V an object $\mathbf{u}+\mathbf{v}$, called the **sum** of $\mathbf{u}$ and $\mathbf{v}$; by **scalar multiplication** we mean a rule for associating with each scalar k and each object $\mathbf{u}$ in V an object $k\mathbf{u}$, called the **scalar multiple** of $\mathbf{u}$ by k. If the following axioms are satisfied by all objects $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars k and m, then we call V a **vector space** and we call the objects in V **vectors**.

1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u}+\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
4. There exists an object in V, called the **zero vector**, that is denoted by $\mathbf{0}$ and has the property that $\mathbf{0}+\mathbf{u}=\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all $\mathbf{u}$ in V.
5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a **negative** of $\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$.
6. If k is any scalar and $\mathbf{u}$ is an object in V, then $k\mathbf{u}$ is in V.
7. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
8. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$
9. $k(m\mathbf{u})=(km)(\mathbf{u})$
10. $1\mathbf{u}=\mathbf{u}$

#6 Determine whether the set equipped with the given operations is a vector space. For those that are not vector spaces, identify the vector space axioms that fail.

The set of all n-tuples of real numbers that have the form $(x,\ x,\ \ldots,\ x)$ with the standard operations on R^n.

Examples of Vector Spaces

1. The simplest vector space is $\{\mathbf{0}\}$.
2. R^n with the usual operations of addition and scalar multiplication of n-tuples
3. R^∞, the set of infinite sequences of numbers with the usual operations of addition and scalar multiplication performed componentwise
4. The set of $m \times n$ matrices with matrix addition and scalar multiplication, denoted M_{mn}
5. The set of real-valued functions $f(x)$ that are defined for all $x \in R$, denoted $F(-\infty, \infty)$ with addition and scalar multiplication
   $(\mathbf{f} + \mathbf{g})(x) = f(x) + g(x)$ and $(k\mathbf{f})(x) = kf(x)$
6. The set of polynomials of degree $\leq n$, denoted P_n

Example: Show why #5 is a vector space.

Example 4.1.7: Not a vector space
Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$. Define $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$ and $k\mathbf{u} = (ku_1, 0)$.

Example 4.1.8: Unusual vector space

Let V be the set of positive real numbers, so $\mathbf{u} = u$ and $\mathbf{v} = v$ are positive real numbers. Define $\mathbf{u} + \mathbf{v} = uv$ and $k\mathbf{u} = u^k$.

Example: Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:
$\mathbf{u} + \mathbf{v} = (u_1 + v_1 + 2,\ u_2 + v_2), k\mathbf{u} = (ku_1, ku_2)$.
a. Show that Axioms 4 and 5 hold.
b. Find all axioms that fail to hold.

Theorem 4.1.1 Let V be a vector space, $\mathbf{u}$ be a vector in V, and k a scalar; then:
a) $0\mathbf{u} = \mathbf{0}$
b) $k\mathbf{0} = \mathbf{0}$
c) $(-1)\mathbf{u} = -\mathbf{u}$
d) If $k\mathbf{u} = \mathbf{0}$, then $k = 0$ or $\mathbf{u} = \mathbf{0}$

Example: Prove that the zero vector in any vector space is unique.