2.1 – Determinants by Cofactor Expansion

Definition: If A is a square matrix, then the **minor of entry $\mathbf{a_{ij}}$** is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the ith row and jth column are deleted from A. The number $(-1)^{i+j}M_{ij}$ is denoted by C_{ij} and is called the **cofactor of entry $\mathbf{a_{ij}}$**.

Example: Find all the minors and cofactors of $A = \begin{bmatrix} 2 & 3 & -7 \\ 4 & -2 & -9 \\ 2 & 5 & 6 \end{bmatrix}$.

Definition: If A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the **determinant of A**, and the sums themselves are called **cofactor expansions of A**. That is,

$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$ (cofactor expansion along the jth column)

$= a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$ (cofactor expansion along the ith row)

Example: Compute the determinant of the matrix A above.

#11 Use the arrow technique of Figure 2.1.1 to evaluate the determinant.

$$\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix}$$

Theorem 2.1.1 If A is an $n \times n$ matrix, then regardless of which row or column of A is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same.

#16 Find all values of λ for which $\det(A) = 0$.

$$A = \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{bmatrix}$$

Theorem 2.1.2 If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then $\det(A)$ is the product of the entries on the main diagonal of the matrix; that is, $\det(A) = a_{11}a_{22} \cdots a_{nn}$.

#31 Evaluate the determinant of the given matrix by inspection.

$$\begin{bmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$