1.6 – More on Linear Systems and Invertible Matrices

Theorem 1.6.1 A system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

Theorem 1.6.2 If A is an invertible $n \times n$ matrix, then for every $n \times 1$ matrix $\mathbf{b}$, the system of equations $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$.

#4 Solve the system by inverting the coefficient matrix and using Theorem 1.6.2.

$$5x_1 + 3x_2 + 2x_3 = 4$$
$$3x_1 + 3x_2 + 2x_3 = 2$$
$$x_2 + x_3 = 5$$

#11 Solve the linear systems. Using the given values for the b's solve the systems together by reducing an appropriate augmented matrix to reduced row echelon form.

$$4x_1 - 7x_2 = b_1$$
$$x_1 + 2x_2 = b_2$$

i. $b_1 = 0, b_2 = 1$
ii. $b_1 = -4, b_2 = 6$
iii. $b_1 = -1, b_2 = 3$
iv. $b_1 = -5, b_2 = 1$

#17 Determine conditions on the b_i's, if any, in order to guarantee that the linear system is consistent.

$$\begin{array}{rrrrl} x_1 & -x_2 & +3x_3 & +2x_4 & = b_1 \\ -2x_1 & +x_2 & +5x_3 & +x_4 & = b_2 \\ -3x_1 & +2x_2 & +2x_3 & -x_4 & = b_3 \\ 4x_1 & -3x_2 & +x_3 & +3x_4 & = b_4 \end{array}$$

Theorem 1.6.4 Equivalent Statements (extends Theorem 1.5.3)
If A is an $n \times n$ matrix, then the following are equivalent.
a) A is invertible.
b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
c) The reduced row echelon form of A is I_n.
d) A is expressible as a product of elementary matrices.
e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

Theorem 1.6.5 Let A and B be square matrices of the same size. If AB is invertible, then A and B must also be invertible.