1.5 – Elementary Matrices and a Method for Finding A^{-1}

Definition: Matrices A and B are said to be **row equivalent** if either (hence each) can be obtained from the other by a sequence of elementary row operations.

Definition: A matrix E is called an **elementary matrix** if it can be obtained from an identity matrix by performing a single elementary row operation.

Theorem 1.5.1 Row Operations by Matrix Multiplication
If the elementary matrix E results from performing a certain row operation on I_m and if A is an $m \times n$ matrix, then the product EA is the matrix that results when this same row operation is performed on A.

#6 An elementary matrix E and a matrix A are given. Identify the row operation corresponding to E and verify that the product EA results from applying the row operation to A.

a. $E = \begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{bmatrix}$

b. $E = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix}$

c. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

#7 Use the following matrices and find an elementary matrix E that satisfies the stated equation.

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix}, B = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix}, C = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix},$$

$$D = \begin{bmatrix} 8 & 1 & 5 \\ -6 & 21 & 3 \\ 3 & 4 & 1 \end{bmatrix}, F = \begin{bmatrix} 8 & 1 & 5 \\ 8 & 1 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

c. $EA = C$

Theorem 1.5.2 Every elementary matrix is invertible, and the inverse is also an elementary matrix.

Theorem 1.5.3 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent, that is, all are true or all false.

a) A is invertible.

b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

c) The reduced row echelon form of A is I_n.

d) A is expressible as a product of elementary matrices.

Use the inversion algorithm to find the inverse of the matrix (if the inverse exists).

#15 $\begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$

#18 $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 2 & 1 & 5 & -3 \end{bmatrix}$

#20 Find the inverse of each of the following 4×4 matrices, where k_1, k_2, k_3, k_4, k are all nonzero.

a. $\begin{bmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{bmatrix}$

b. $\begin{bmatrix} k & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & k \end{bmatrix}$

#27 Show that the matrices A and B are row equivalent by finding a sequence of elementary row operations that produces B from A, and then use that result to find a matrix C such that $CA = B$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 9 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 1 & 1 & 4 \end{bmatrix}$$

#23 Express the matrix and its inverse as products of elementary matrices.

$$\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix}$$

#33 Prove that if B is obtained from A by performing a sequence of elementary row operations, then there is a second sequence of elementary row operations, which when applied to B recovers A.