Tests for Convergence Summary

Test for Divergence: If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

The Integral Test: Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum\limits_{n=1}^{\infty} a_n$ is convergent $\iff \int_1^{\infty} f(x)\,dx$ is convergent.

The Comparison Test: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
ii) If $\sum b_n$ is divergent and $b_n \leq a_n$ for all n, then $\sum a_n$ is also divergent.

The Limit Comparison Test: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c$, where c is a positive, finite number, then either both series converge or both series diverge.

If the limit is zero and $\sum b_n$ is convergent, then so is $\sum a_n$.
If the limit is ∞ and $\sum b_n$ is divergent, then so is $\sum a_n$.
Two special cases:

- If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ AND if $\sum b_n$ converges, then $\sum a_n$ also converges.
- If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$ AND if $\sum b_n$ diverges, then $\sum a_n$ also diverges.

The Alternating Series Test: If the alternating series
$\sum\limits_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$, where $b_n > 0$, satisfies

i) $b_{n+1} \leq b_n$ for all n
ii) $\lim\limits_{n\to\infty} b_n = 0$

then the series is convergent.

The Ratio Test:

i) If $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L < 1$, then the series $\sum a_n$ is convergent
ii) If $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L > 1$ or is ∞, then the series $\sum a_n$ is divergent
iii) If $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$, the Ratio Test is inconclusive

The Root Test:

i) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum a_n$ is convergent
ii) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or is ∞, then the series $\sum a_n$ is divergent
ii) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive

The remainder of a partial sum and estimating sums: Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent with sum s. Then if S_n is a partial sum,

- $R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$ is the remainder in approximating s
- $\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_n^{\infty} f(x)\,dx$
- $s_n + \int_{n+1}^{\infty} f(x)\,dx \leq s \leq s_n + \int_n^{\infty} f(x)\,dx$

The Alternating Series Estimation Theorem: If $S = \sum\limits_{n=1}^{\infty} (-1)^{n-1} b_n$ is a series that converges by the Alternating Series Test, then $|R_n| \leq b_{n+1}$.