

11.10 - Taylor and Maclaurin Series

11.10a

[illegible]

Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Maclaurin series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

11.10b

Power series representations of more functions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

11.10c

Taylor polynomials

$$T_n(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x-a)^i$$

Theorem (statement):

Let $R_n(x) = f(x) - T_n(x)$. If $\lim_{n \rightarrow \infty} R_n(x) = 0$, then f is equal to the sum of its Taylor series

Taylor's Inequality

If $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$, then the remainder $R_n(x)$ of the Taylor series of f satisfies $|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1}$ for $|x - a| \leq d$.

11.10d

The Binomial Series

$$\begin{aligned}(1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n \\ &= 1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots + \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n + \cdots\end{aligned}$$

[illegible]

11.10e

Exercises:

Use the definition of a Taylor series to find the first four nonzero terms of the series for $f(x)$ centered at the given value of a .

$$f(x) = \ln x, a = 1$$

11.10f

Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. [Assume that f has a power series expansion. Do not show that $R_n(x) \rightarrow 0$.] Also find the associated radius of convergence.

$$f(x) = e^{-2x}$$

11.10g

Use the binomial series to expand the function as a power series. State the radius of convergence.

$$\sqrt[3]{8+x}$$

11.10h

Use a known Maclaurin series to obtain the Maclaurin series for the given function.

$$f(x) = e^{3x} - e^{2x}$$

$$f(x) = x^2 \ln(1 + x^3)$$

Use series to evaluate the limit.

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{1 + x - e^x}$$

Find the sum of the series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n} (2n)!}$$

11.10i

Use multiplication or division of power series to find the first three nonzero terms in the Maclaurin series for each function.

$$y = e^x \ln(1 + x)$$

$$y = \sec x$$

11.10j

Evaluate the indefinite integral as an infinite series.

$$\int x^2 \sin(x^2) dx$$
