10.4 - Areas and Lengths in Polar Coordinates

10.4a

Area in polar coordinates

$$A = \int_a^b \frac{1}{2} [f(\theta)]^2 \ d\theta$$

Between two curves f and g ($f \geq g$): $A = \frac{1}{2} \int_a^b \left\{ [f(\theta)]^2 - [g(\theta)]^2 \right\} \ d\theta$

Arc length

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Exercises:

10.4b

Graph the curve and find the area that it encloses.

$r = 3 - 2\cos 4\theta$

10.4c

Find the area of the region that lies inside the first curve and outside the second curve.

$r = 3\sin\theta,\ r = 2 - \sin\theta$

10.4d

Find the area of the region that lies inside both curves.

$r = \sin 2\theta$, $r = \cos 2\theta$

10.4e

Find the length of the polar curve.

$r = 5^{\theta}, 0 \leq \theta \leq 2\pi$

10.4f

Tangents to a polar curve: if $r = f(\theta)$ and $x = r\cos\theta$, $y = r\sin\theta$, then

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta \text{ (by the product rule)}$$

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \text{ (product rule again)}$$

$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Horizontal tangents occur when $\dfrac{dy}{d\theta} = 0$ (if $\dfrac{dy}{dx} \neq 0$)

Vertical tangents occur when $\dfrac{dx}{d\theta} = 0$ (if $\dfrac{dx}{dy} \neq 0$)

Note: if both $\dfrac{dy}{d\theta} = 0$ and $\dfrac{dx}{d\theta} = 0$, we use l'Hospital's Rule

Exercises:

Find the slope of the tangent line to the given polar curve at the point specified by the value of θ.

$r = 2 + \sin 3\theta$, $\theta = \dfrac{\pi}{4}$

10.4g

Find the points on the given curve where the tangent line is horizontal or vertical.

$r = 1 - \sin \theta$