10.2 - Calculus with Parametric Curves

10.2a

Calculus with curves defined by parametric equations $x = f(t)$, $y = g(t)$

Derivatives

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}, \text{ if } \frac{dx}{dt} \neq 0$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{g'(t)}{f'(t)}\right)}{\frac{dx}{dt}}$$

Area

$$A = \int_a^b y\, dx = \int_\alpha^\beta g(t)f'(t)\, dt$$

Arc length

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_\alpha^\beta \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}\, dt$$

Area of a surface of revolution

By rotating about the x-axis: $S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

By rotating about the y-axis: $S = \int_{\alpha}^{\beta} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Exercises:

10.2b

Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter.

$x = e^t \sin \pi t,\ y = e^{2t},\ t = 0$

10.2c

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. For which values of t is the curve concave upward?

$x = t^2 + 1,\ y = e^t + 1$

10.2d

Find points on the curve where the tangent is horizontal or vertical.

$x = t^3 - 3t,\ y = t^3 - 3t^2$

10.2e

Find the area of the region enclosed by the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$. Use a graphing device to graph the curve for $0 \leq \theta \leq 2\pi$ and an a-value of your choosing.

Example: What about $x = a\cos\theta$, $y = a\sin\theta$, $0 \leq \theta \leq 2\pi$?

10.2f

Find the exact length of the curve.

$x = 3\cos t - \cos 3t$, $y = 3\sin t - \sin 3t$, $0 \leq t \leq \pi$

Example: What about $x = a\cos\theta$, $y = a\sin\theta$, $0 \leq \theta \leq 2\pi$?

10.2g

Find the exact area of the surface obtained by rotating the given curve about the y-axis.

$x = 3t^2$, $y = 2t^3$, $0 \leq t \leq 5$