6.8 - Indeterminate Forms and l'Hospital's Rule

6.8a

Consider $\lim_{x \to \infty} \frac{\ln x}{x} \to \frac{\infty}{\infty}$

and $\lim_{x \to 0} \frac{\sin x}{x} \to \frac{0}{0}$

l'Hôspital's Rule for indeterminate forms

Suppose that f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} = g(x) = 0$
or that $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} = g(x) = \pm\infty$

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ if the limit on the right-hand side exists or is ∞ or $-\infty$

Indeterminate forms

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 0^0, ∞^0, and 1^∞

Turn this into

Note: $0^\infty = 0$
(not indeterminate)

l'Hospital's Rule is a tool for finding limits of expressions that yield indeterminate forms.

Exercises:

Find the limit. Use l'Hôspital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hôspital's Rule doesn't apply, explain why.

6.8b

$$\lim_{x\to 0} \frac{x^2}{1-\cos x}$$

Check: $\frac{0}{0}$ ✓

$\overset{H}{=} \lim\limits_{x\to 0} \frac{2x}{\sin x}$ $\qquad \frac{0}{0}$ still

$\overset{H}{=} \lim\limits_{x\to 0} \frac{2}{\cos x} = 2$

$$\lim_{\theta\to \pi} \frac{1+\cos\theta}{1-\cos\theta}$$

$\frac{0}{2} = 0$

what if: $\overset{H}{=} \lim\limits_{\theta\to\pi} \frac{-\sin\theta}{\sin\theta} = -1$ ✓ incorrect.

l'Hopital's Rule did not apply.

This is not an indeterminate form.

$$\lim_{x\to \infty} \frac{\ln\sqrt{x}}{x^2}$$

$\frac{\infty}{\infty}$ ✓

$= \lim\limits_{x\to\infty} \frac{\frac{1}{2}\ln x}{x^2} \overset{H}{=} \frac{1}{2}\lim\limits_{x\to\infty} \frac{1/x}{2x} = \frac{1}{4}\lim\limits_{x\to\infty}\frac{1}{x^2}$

$= 0$

$$\lim_{x\to 0} \frac{\sinh x - x}{x^3}$$

$\frac{0}{0}$ ✓

Recall $\sinh x = \frac{e^x - e^{-x}}{2}$

$$\overset{H}{=} \lim_{x\to 0} \frac{\cosh x - 1}{3x^2} \qquad \frac{0}{0} \text{ still} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\overset{H}{=} \lim_{x\to 0} \frac{\sinh x}{6x} \overset{H}{=} \lim_{x\to 0} \frac{\cosh x}{6} = \frac{1}{6}$$

6.8c

$$\lim_{t\to 0} \frac{8^t - 5^t}{t} \qquad \frac{1-1}{0} = \frac{0}{0} \checkmark$$

$$\overset{H}{=} \lim_{t\to 0} \frac{\ln 8 \cdot 8^t - \ln 5 \cdot 5^t}{1} = \ln 8 - \ln 5$$

$$= \ln 8/5$$

Either is fine

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}$$

$$\lim_{x\to a^+} \frac{\cos x \ln(x-a)}{\ln(e^x - e^a)} \qquad \frac{\cos a\,(-\infty)}{(-\infty)} \to \frac{\infty}{\infty} \checkmark$$

$$= \lim_{x\to a^+} \cos x \lim_{x\to a^+} \frac{\ln(x-a)}{\ln(e^x - e^a)} \overset{H}{=} \cos a \lim_{x\to a^+} \frac{\frac{1}{x-a}}{\frac{e^x}{e^x - e^a}}$$

$$= \cos a \lim_{x\to a^+} \frac{e^x - e^a}{e^x(x-a)} = \cos a \lim_{x\to a^+} \frac{1}{e^x} \lim_{x\to a^+} \frac{e^x - e^a}{x-a}$$

$$= \cos a \left(\frac{1}{e^a}\right)(e^a) = \cos a$$

$$\frac{d}{dx}(e^x)\Big|_{x=a}$$

Recall: $\lim_{x\to a} [f(x) g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$,

provided both limits exist.

6.8d

$$\lim_{x\to-\infty} x \ln\left(1-\frac{1}{x}\right)$$

Check: $-\infty(0)$ $\frac{0}{0}$

$$= \lim_{x\to-\infty} \frac{\ln\left(1-\frac{1}{x}\right)}{\frac{1}{x}} \overset{H}{=} \lim_{x\to-\infty} \frac{\frac{1/x^2}{1-\frac{1}{x}}}{-1/x^2} \quad \frac{x^2}{x^2}$$

$$= \lim_{x\to-\infty}\left(-\frac{1}{1-\frac{1}{x}}\right) = -1$$

$$\lim_{x\to 1^+}\left[\ln\left(x^7-1\right)-\ln\left(x^5-1\right)\right]$$

Check: $-\infty+\infty$

$$= \lim_{x\to1^+} \ln \frac{x^7-1}{x^5-1} = \ln \lim_{x\to1^+} \frac{x^7-1}{x^5-1}$$

$$\overset{H}{=} \ln \lim_{x\to1^+} \frac{7x^6}{5x^4} = \ln \lim_{x\to1} \frac{7x^2}{5} = \ln\frac{7}{5}$$

6.8e

$$\lim_{x\to0^+} (\tan 2x)^x$$

6.8f

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = L$$

Check: 1^{∞}

$$\lim_{x\to\infty} \ln\left(1+\frac{a}{x}\right)^{bx} = \ln L$$

$$\lim_{x\to\infty} bx \ln\left(1+\frac{a}{x}\right) = \ln L$$

$\infty \cdot 0$

$$b\lim_{x\to\infty} \frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{x}} = \ln L$$

$b \lim\limits_{x\to\infty} \dfrac{\cancel{-} a/\cancel{x^2}}{\dfrac{1+a/x}{\cancel{-}\cancel{1/x^2}}} \stackrel{H}{=} \ln L$

$ab \lim\limits_{x\to\infty} \dfrac{1}{1+\frac{a}{x}} = \ln L$

$ab = \ln L \Rightarrow L = e^{ab}$

Finance - interest: $A = P\left(1+\frac{r}{n}\right)^{nt} \xrightarrow[n\to\infty]{} Pe^{rt}$

Here: $\lim\limits_{x\to\infty} \left(1+\frac{a}{x}\right)^{bx}$

6.8g

Proof:

l'Hôspital's Rule

Suppose that f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

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Then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ if the limit on the right-hand side exists or is ∞ or $-\infty$

Indeterminate forms

$\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, \infty-\infty, 0^0, \infty^0$, and 1^∞

$\frac{0}{0}$

1. $\lim\limits_{x\to0}\dfrac{\tan 3x}{\sin 2x} \overset{H}{=} \lim\limits_{x\to0}\dfrac{3\sec^2 3x}{2\cos 2x} = \dfrac{3}{2}$

$\dfrac{e^x}{x^n} \xrightarrow[\text{as } x\to\infty]{} \infty \qquad \dfrac{x^n}{e^x} \xrightarrow[\text{as } x\to\infty]{} 0$

$\dfrac{\infty}{\infty}$

2. $\lim\limits_{u\to\infty}\dfrac{e^{\frac{u}{10}}}{u^3} \overset{H}{=} \lim\limits_{u\to\infty}\dfrac{\frac{1}{10}e^{u/10}}{3u^2} \overset{H}{=} \lim\limits_{u\to\infty}\dfrac{1/100\, e^{u/10}}{6u}$

$\overset{H}{=} \lim\limits_{u\to\infty}\dfrac{1/1000\, e^{u/10}}{6} = \infty$

$\sin^2 x+\cos^2 x=1 \Rightarrow \tan^2 x+1=\sec^2 x \Rightarrow \tan^2 x=\sec^2 x-1$

$\dfrac{0}{0} \qquad \dfrac{0}{0}$

3. $\lim\limits_{x\to0}\dfrac{x-\sin x}{x-\tan x} \overset{H}{=} \lim\limits_{x\to0}\dfrac{1-\cos x}{1-\sec^2 x}$

$= \lim\limits_{x\to0}\dfrac{\cos x-1}{\tan^2 x} \overset{H}{=} \lim\limits_{x\to0}\dfrac{-\sin x}{2\frac{\sin x}{\cos x}\sec^2 x}$

$= \lim\limits_{x\to0}\dfrac{-\cancel{\sin x}\,\frac{\cos x}{\cancel{\sin x}}}{2\sec^2 x} = -\dfrac{1}{2}$

$\infty \cdot 0$

4. $\lim\limits_{x\to\infty} x^{\frac{3}{2}} \sin\left(\frac{1}{x}\right) = \lim\limits_{x\to\infty} \dfrac{\sin\frac{1}{x}}{\frac{1}{x^{3/2}}} \leftarrow x^{-3/2}$

$$\overset{H}{=} \lim_{x\to\infty} \frac{\cos\frac{1}{x}\cdot\left(\cancel{-}\frac{1}{x^2}\right)}{\cancel{-}\frac{3}{2x^{5/2}}} \qquad \frac{x^{5/2}}{x^{5/2}}$$

$$= \frac{2}{3}\lim_{x\to\infty} \cos\frac{1}{x}\sqrt{x} = \infty$$