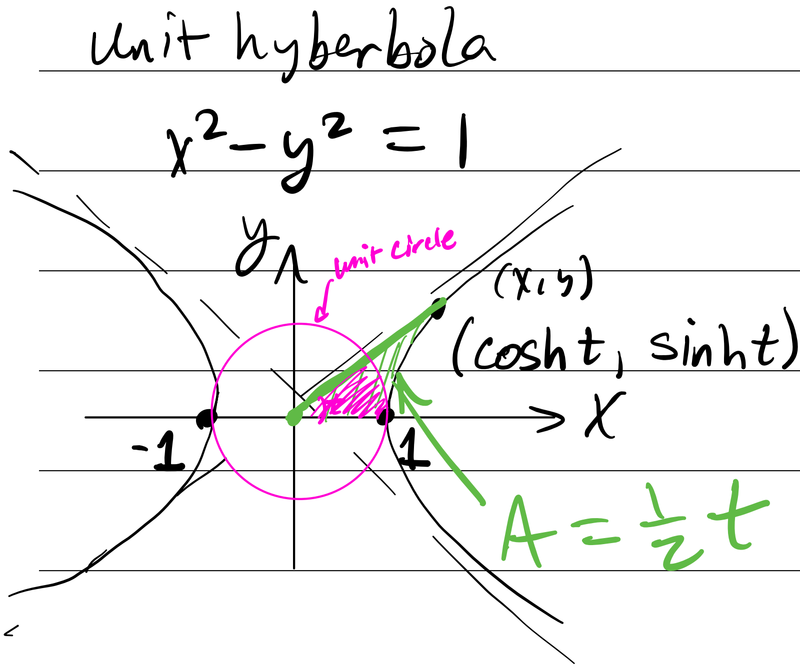
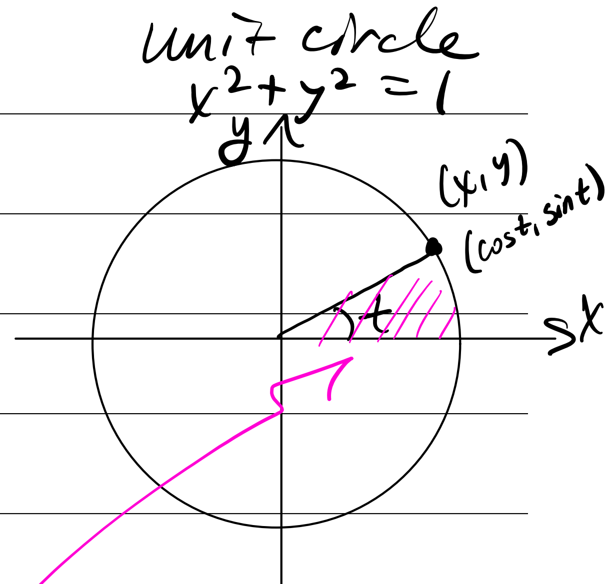
6.7 - Hyperbolic Functions

6.7a

Definitions: (hyperbolic functions)





x: hyperbolic cosine

y: hyperbolic sine

A_{sector} : $A = \frac{1}{2} r^2 \theta$

unit circle: $r = 1, \theta = t$

$A = \frac{1}{2} t$

$\sinh x = \dfrac{e^x - e^{-x}}{2}$ $\operatorname{csch} x = \dfrac{1}{\sinh x}$

$\cosh x = \dfrac{e^x + e^{-x}}{2}$ $\operatorname{sech} x = \dfrac{1}{\cosh x}$

$\tanh x = \dfrac{\sinh x}{\cosh x}$ $\coth x = \dfrac{\cosh x}{\sinh x}$

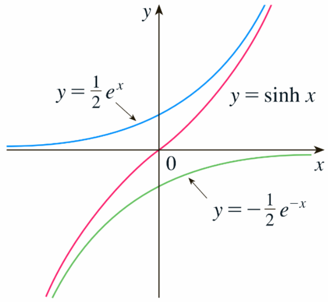


FIGURE 1
$y = \sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

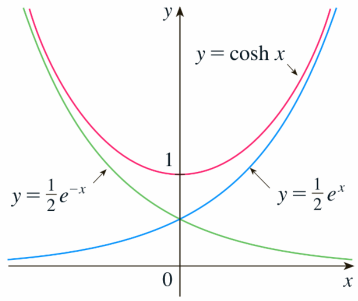


FIGURE 2
$y = \cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

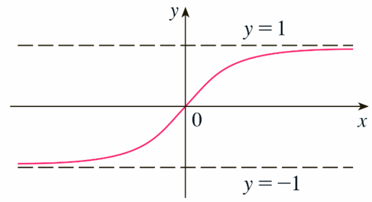


FIGURE 3
$y = \tanh x$

6.7b

Exercises:

Find the numerical value of each expression.

a) sinh 4 b) sinh (ln 4)

a) $\sinh 4 = \dfrac{e^4 - e^{-4}}{2} \approx 27.3$

b) $\sinh(\ln 4) = \dfrac{e^{\ln 4} - e^{-\ln 4}}{2} = \dfrac{4 - \frac{1}{4}}{2}$

$= \dfrac{15}{8}$

Prove the identity.

$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

RHS: $\dfrac{e^x + e^{-x}}{2} \cdot \dfrac{e^y + e^{-y}}{2} + \dfrac{e^x - e^{-x}}{2} \cdot \dfrac{e^y - e^{-y}}{2}$

$= \dfrac{e^{x+y} + \cancel{2e^{-xy}} + e^{-(x+y)}}{4} + \dfrac{e^{x+y} - \cancel{2e^{-xy}} + e^{-(x+y)}}{4}$

$$= \frac{e^{x+y} + e^{-(x+y)}}{2} = \cosh(x+y)$$

Ex: $\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)$

$$= \frac{e^x + e^{-x}}{2} = \cosh x$$

6.7c

Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \text{sech}^2 x$$

$$\frac{d}{dx}(sechx) = -\text{sech}x \tanh x$$

$$\frac{d}{dx}(\text{csch}x) = -\text{csch}x \coth x$$

$$\frac{d}{dx}(\coth x) = -\text{csch}^2 x$$

Note that reversing these gives integration rules

Exercises:

Find the derivative. Simplify where possible.

$F(t) = \ln(\sinh t)$

$$F'(t) = \frac{\cosh t}{\sinh t} = \coth t$$

Evaluate the integral.

$$\int \frac{\operatorname{sech}^2 x}{2+\tanh x}\,dx = \int \frac{du}{2+u}$$

$u = \tanh x \Rightarrow du = \operatorname{sech}^2 x\, dx$

$= \ln|2+u| + C = \ln(2+\tanh x + C$

6.7d

Inverse hyperbolic functions

Consider $y = \cosh^{-1} x$ so $x = \cosh y$

$\Rightarrow x = \frac{e^y + e^{-y}}{2} \Rightarrow 2x = e^y + e^{-y}$

$\Rightarrow e^y(e^y - 2x + e^{-y}) = (0)\, e^y$

$e^{2y} - 2xe^y + 1 = 0$ Form $a(\ )^2 + b(\ ) + c = 0$

quadratic in e^y

$e^y = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \cancel{\pm} \sqrt{x^2-1}$

by convention

$\cosh^{-1} x = y = \ln\left(x + \sqrt{x^2-1}\right)$

$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}),\ x \in \mathbb{R}$

$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}),\ x \geq 1$

$\tanh^{-1} x = \dfrac{1}{2} \ln\left(\frac{1+x}{1-x}\right),\ -1 < x < 1$

Derivatives of inverse hyperbolic functions

$$\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = -\frac{1}{|x|\sqrt{x^2+1}}$$

$$\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}} \qquad \frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = -\frac{1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2},\ |x| < 1 \qquad \frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2},\ |x| > 1$$

Exercise:

Evaluate the integral.

$$\frac{1}{4}\int_0^1 \frac{4 \cdot 1}{\sqrt{16t^2+1}}\, dt$$

$u = 4t \Rightarrow du = 4dt$

Limits

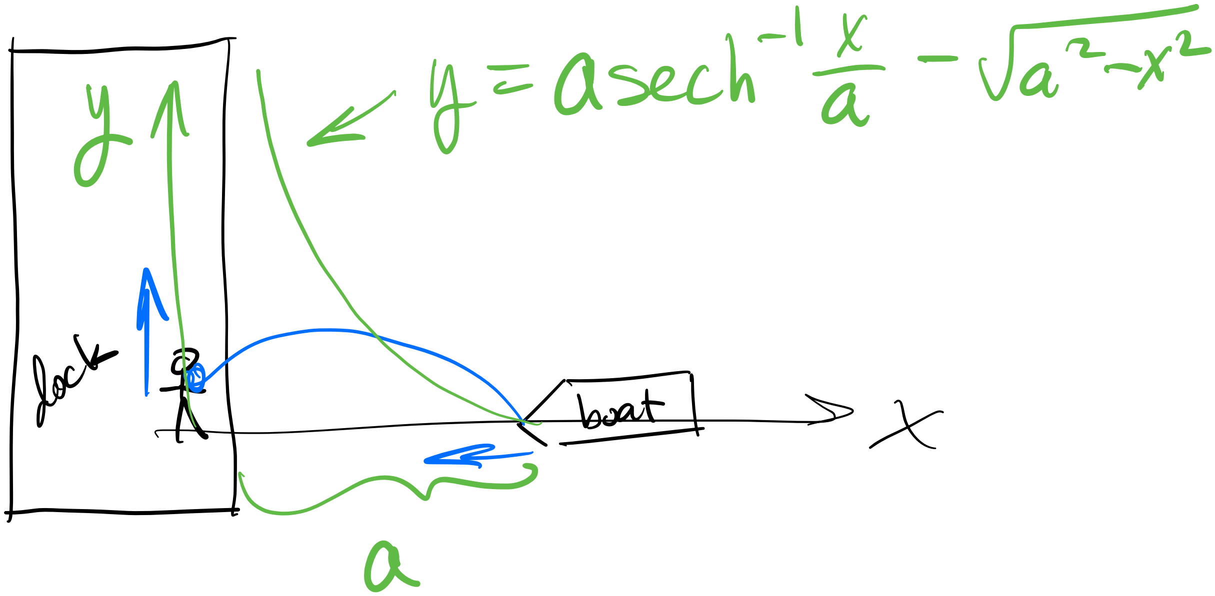
t	$u = 4t$
1	4
0	0

$$\frac{1}{4}\int_0^4 \frac{1}{\sqrt{u^2+1}}\, du$$

$$= \frac{1}{4}\sinh^{-1} u \Big|_0^4 = \frac{1}{4}\sinh^{-1} 4 - \frac{1}{4}\sinh^{-1} 0$$

$\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$

$= \frac{1}{4}\ln(4+\sqrt{17})$



$y = a\,\mathrm{sech}^{-1}\frac{x}{a} - \sqrt{a^2 - x^2}$
y
dock
boat
x
a