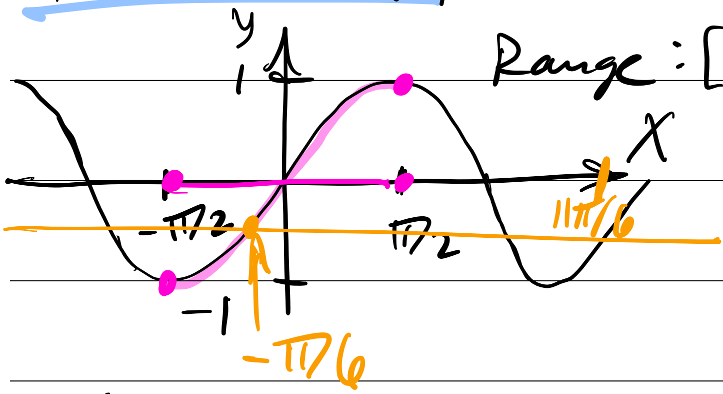
6.6 - Inverse Trigonometric Functions

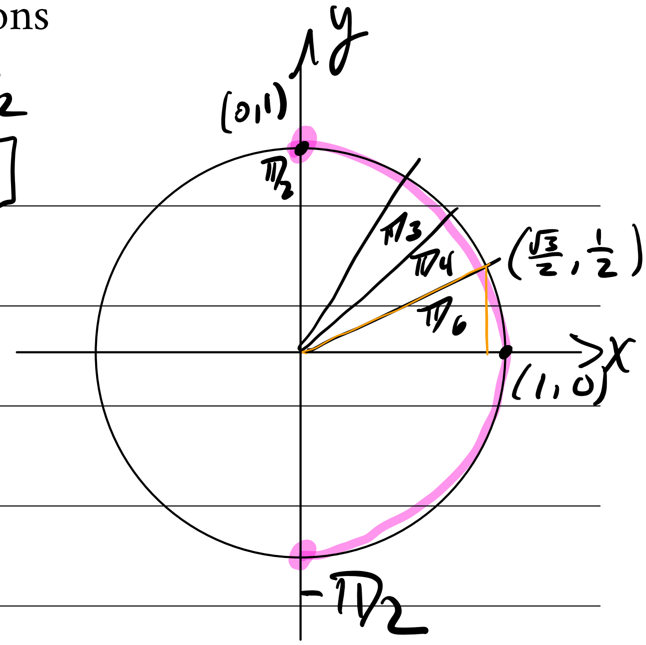
6.6a

A brief review of inverse trigonometric functions

$f(x) = \sin x, \quad -\pi/2 \le x \le \pi/2$

Range: $[-1, 1]$

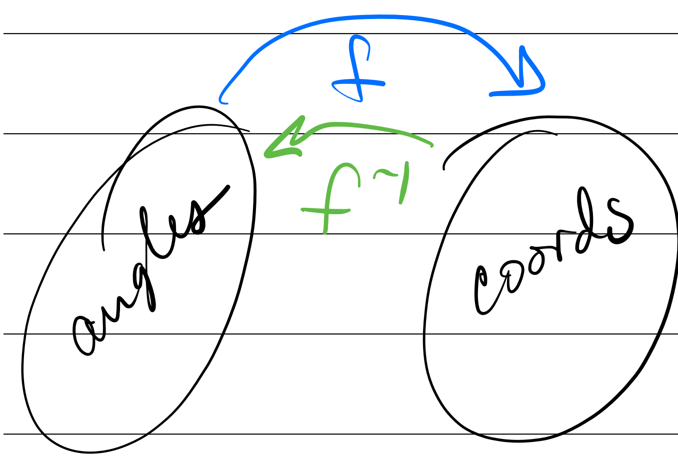




$f^{-1}(x) = \sin^{-1} x$

Dom: $[-1, 1]$

Range: $[-\pi/2, \pi/2]$



Dom f

Range f

$\sin \pi/6 = \frac{1}{2}$

$\sin \frac{11\pi}{6} = -\frac{1}{2}$

$\sin^{-1} \frac{1}{2} = \arcsin \frac{1}{2}$

$\sin^{-1}(-\frac{1}{2}) = -\pi/6$

"What is the angle in $[-\pi/2, \pi/2]$ whose y-coord is $\frac{1}{2}$?"

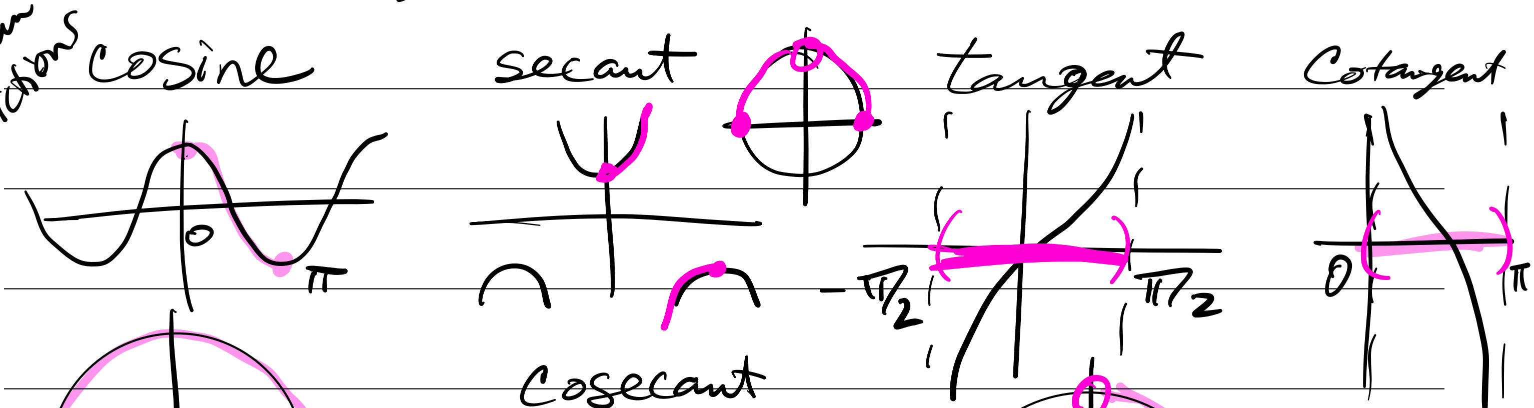
Ex: $\sin^{-1}(\sin 2\pi/3) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

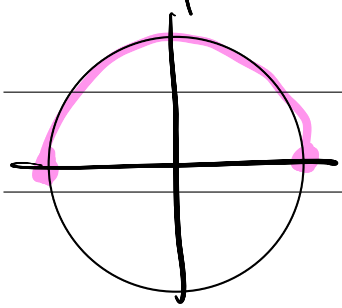
$= \pi/3$

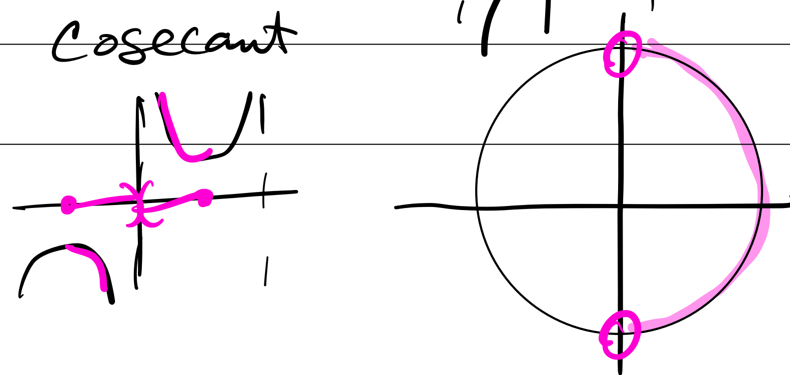
$\sin\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) = -\frac{1}{2}$

$\sin^{-1}(2)$ is not defined

Domain restrictions







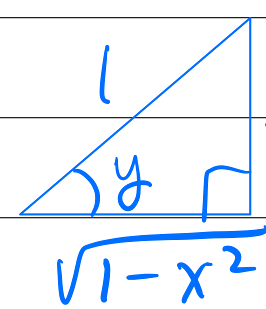
6.6b

Calculus of inverse trigonometric functions

Consider $x = \sin y \iff y = \sin^{-1} x$

differentiate WRT x: $1 = \cos y \, y'$

$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$



$\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1-x^2}}$,

$-1 < x < 1$

$\sin y = x$

$\cos y = \sqrt{1-x^2}$

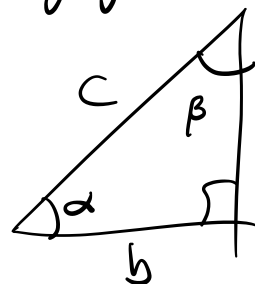
$$\int -\frac{1}{\sqrt{1-x^2}}\,dx = -\sin^{-1}x + C$$

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \Longrightarrow \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$$

$x = \cos y \Rightarrow 1 = -\sin y \, y'$

$$\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{x^2+1} \Longrightarrow \int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C$$



$\sin\alpha = \frac{a}{c} = \cos\beta$

$\alpha + \beta = \pi/2$

$$\frac{d}{dx}\left(\cot^{-1}x\right) = -\frac{1}{x^2+1}$$

$\sin^{-1}\frac{a}{c} + \cos^{-1}\frac{a}{c} = \alpha+\beta = \pi/2$

$$\frac{d}{dx}\left(\sec^{-1}x\right) = \frac{1}{x\sqrt{x^2-1}} \Longrightarrow \int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + C$$

$\sin^{-1}\frac{a}{c} = \boxed{\pi/2 - \cos^{-1}\frac{a}{c}}$

$\sin^{-1}x = \pi/2 - \cos^{-1}x$

$$\frac{d}{dx}\left(\csc^{-1}x\right) = -\frac{1}{x\sqrt{x^2-1}}$$

$\frac{d}{dx}(\sin^{-1}x) = -\frac{d}{dx}(\cos^{-1}x)$

Note: the formula for the derivative of the inverse secant assumes a restricted domain for the secant of $\{x|\ 0 < x < \frac{\pi}{2}\ or\ \pi < x < \frac{3\pi}{2}\}$. If instead the restricted domain is $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$, then the differentiation rule becomes $\frac{d}{dx}\left(\sec^{-1}x\right) = \frac{1}{|x|\sqrt{x^2-1}}$.

6.6c

Exercises:

Find the derivative of the function. Simplify where possible.

$g(x) = \arccos\sqrt{x}$

Chain Rule

$g'(x) = -\frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$

$g'(x) = -\frac{1}{2\sqrt{x-x}}$

$y = \cos^{-1}\left(\sin^{-1} t\right)$

$$y' = -\frac{1}{\sqrt{1-(\sin^{-1}t)^2}}\left(\frac{1}{\sqrt{1-t^2}}\right)$$

$$y' = -\frac{1}{\sqrt{1-(\sin^{-1}t)^2}\sqrt{1-t^2}}$$

$R(t) = \arcsin\left(\frac{1}{t}\right)$

$$\sqrt{50} = 5\sqrt{2} \qquad 5\sqrt{2} = \sqrt{25\cdot 2}$$

$$R'(t) = \frac{1}{\sqrt{1-1/t^2}}\left(-\frac{1}{t^2}\right) = \frac{1}{\sqrt{1-1/t^2}}\cdot -\left(\frac{1}{\sqrt{t^4}}\right)$$

$$R'(t) = -\frac{1}{\sqrt{t^4-t^2}} = -\frac{1}{|t|\sqrt{t^2-1}}$$

$$\sqrt{t^2(t^2-1)} = \underbrace{\sqrt{t^2}}_{|t| \text{ by def.}}\sqrt{t^2-1}$$

6.6d

$y = \arctan\sqrt{\dfrac{1-x}{1+x}}$

6.6e

Evaluate the integral

$$\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{6}{\sqrt{1-p^2}}\,dp \quad = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{6}{\sqrt{1-p^2}}\,dp$$

$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

What is the angle whose sine is $\frac{1}{\sqrt{2}}$?

$$= 6 \sin^{-1} p \Big|_{-1/\sqrt{2}}^{1/\sqrt{2}} = 6\left(\sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right)$$

$$= 6\left(\pi/4 - (-\pi/4)\right)$$

$$= 3\pi$$

4.5 - The Substitution Rule

$$\frac{1}{4}\int_0^{\frac{\sqrt{3}}{4}} \frac{4\,dx}{1+16x^2}$$

$$\frac{1}{32}\int \frac{32x}{1+16x^2}\,dx = \frac{1}{32}\ln(1+16x^2)+C$$

If $u = 16x^2$, then $du = 32x\,dx$ ✗

$u = 4x \Rightarrow du = 4\,dx$

x	$u=4x$
$\sqrt{3}/4$	$\sqrt{3}$
0	0

$$= \frac{1}{4}\int_0^{\sqrt{3}} \frac{1}{1+u^2}\,du = \frac{1}{4}\tan^{-1}u\Big|_0^{\sqrt{3}} = \pi/12$$