6.5 - Exponential Growth and Decay

6.5a

Exponential growth and decay equation

$P(t) = P_0 e^{kt}$, where P_0 is the initial population and k is the constant relative growth rate

growth/decay rate is proportional to a quantity

$$\frac{dP}{dt} = kP \quad \text{OR} \quad \frac{dP/dt}{P} = k$$

$$\int \frac{dP}{P} = \int k\,dt \Rightarrow \ln|P| = kt + C$$

$e \qquad e \qquad x^2 \cdot x^3 = x^5$

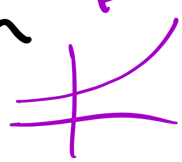
$$|P| = e^{kt+C} = e^{kt}e^{C} = C_1 e^{kt}$$

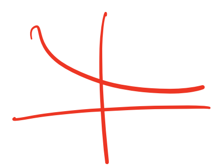
$$P(t) = P_0 e^{kt} \qquad A(t) = A_0 e^{kt}$$

initial population

growth if $k > 0$

decay if $k < 0$





Exercise:

A bacteria culture grows with constant relative growth rate. The bacteria count was 400 after 2 hours and 25,600 after 6 hours.

$\to P(t) = P_0 e^{kt}$

When $t = 2$, $P = 400$ $\qquad$ when $t = 6$, $P = 25{,}600$

$$25{,}600 = \cancel{P_0} e^{6k}$$

$$\div \qquad \div$$

$$400 = \cancel{P_0} e^{2k}$$

$$64 = e^{4k}$$

$\ln 64 = 4k \Rightarrow k = \frac{1}{4}\ln 64$

$k \approx 1.0397$

$400 = P_0 e^{2(1.0397)}$

$P_0 \approx 50.002$

$P'(t) = 1.04(50)e^{1.04t}$

$P'(4.5) = 1.04(50)e^{1.04(4.5)}$

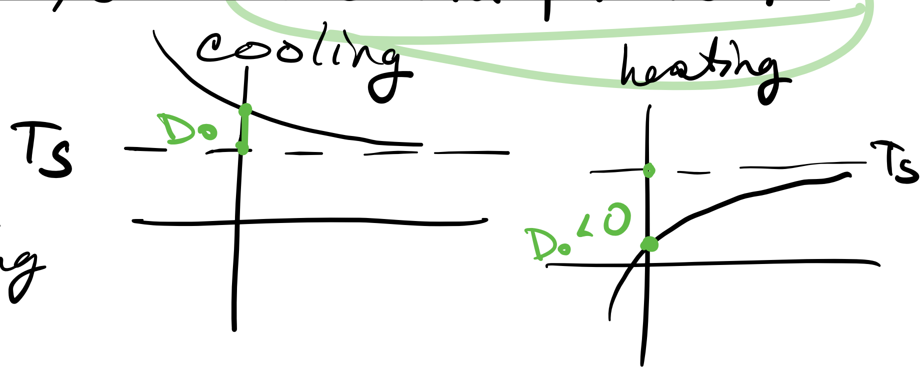
a) What is the relative growth rate? Express your answer as a percentage.
b) What was the initial size of the culture?
c) Find an expression for the number of bacteria after t hours.
d) Find the rate of growth after 4.5 hours. $\rightarrow \frac{dP}{dt}$

a) $\sim 104\%$

b) $P_0 = 50$ bacteria

c) $P(t) = 50e^{1.04t}$

d) $P'(4.5) \approx 5{,}604$ bacteria per hour



6.5b

Newton's Law of Cooling /heating

$T(t) = D_0 e^{kt} + T_s$, where D_0 is the initial temperature difference and T_s is the surrounding temperature

Rate of temp change is proportional to temp. difference

$$\frac{dT}{dt} = k(T - T_s) \Rightarrow \frac{dT}{T - T_s} = k\,dt$$

$$\ln|T - T_s| = kt + C$$

$$|T - T_s| = c_1 e^{kt}$$

$$T - T_s = D_0 e^{kt}$$

surrounding temp

$$T(t) = D_0 e^{kt} + T_s \qquad k < 0$$

initial temp. difference

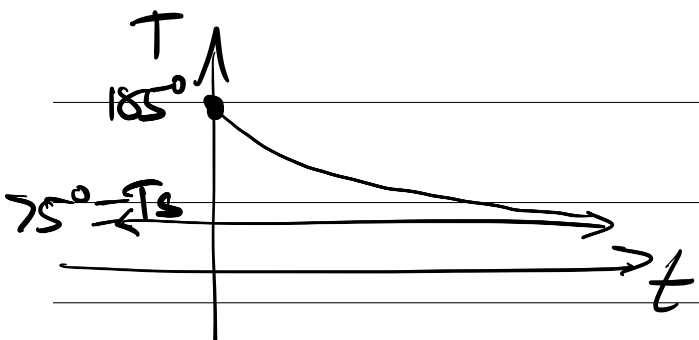
Exercise:

A roast turkey is removed from an oven when its temperature has reached 185° F and is placed on a table in a room where the ambient temperature is 75° F. $\leftarrow T_s$

a) If the temperature of the turkey is 150° F after half an hour, what is the temperature after 45 minutes?

when $t = 30$, $T = 150$

b) When will the turkey have cooled to 100° F?



$$D_0 = 185° - 75° = 110°$$

$$T(t) = 110 e^{kt} + 75$$

$$150 = 110 e^{30k} + 75 \Rightarrow \frac{75}{110} = e^{30k}$$

$$k = \frac{1}{30} \ln \frac{75}{110} \approx -0.01277$$

a) $T(t) = 110e^{-0.013t} + 75$

$T(45) = 110e^{-0.013(45)} + 75 \approx 136°F$

b) $100 = 110e^{-0.013t} + 75$

$t = -\frac{1}{0.013}\ln\frac{25}{110} \approx 114$ min

Doubling time (growth) $k>0$ & half-life (decay) $k<0$

$P(t) = P_0 e^{kt}$

$P_0 \longrightarrow 2P_0$

initial

$A_0 \rightarrow \frac{1}{2}A_0$

Time to double:

$2\cancel{P_0} = \cancel{P_0} e^{kt}$

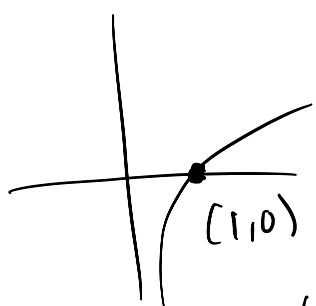
$\ln 2 = kt$

$t = \frac{\ln 2}{k}$

$\frac{1}{2}\cancel{A_0} = \cancel{A_0} e^{kt}$

$\ln\frac{1}{2} = kt$

$t = \frac{\ln 1/2}{k}$



half-life of C-14 is

~ 5730 years

$\lambda = 5730$

$$\frac{1}{2}A_0 = A_0 e^{5730k}$$