6.4 - Derivatives of Logarithmic Functions

6.4a

Calculus of the natural logarithmic function

Recall that $y = \ln x \iff e^y = x$.

Using implicit differentiation, we get

$$x = e^y$$

diff. WRT x: $1 = e^y \frac{dy}{dx}$

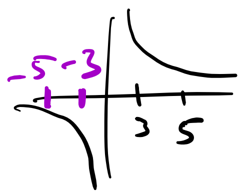
$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x} \Rightarrow \frac{d}{dx}[\ln x] = \frac{1}{x}$$

If $x<0$, then $\frac{d}{dx}[\ln(-x)] = \frac{1}{-x}(-1) = \frac{1}{x}$

Chain Rule

$$\frac{d}{dx}[\ln|x|] = \frac{1}{x},\ x \neq 0$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$



$$\int_3^5 \frac{1}{x}dx = \ln x\Big|_3^5 = \ln 5 - \ln 3$$

$$\int_{-5}^{-3} \frac{1}{x}dx = \ln|x|\Big|_{-5}^{-3} = \ln 3 - \ln 5$$

Power Rule for integrating:

$$\int x^n dx = \begin{cases} \frac{1}{n+1}x^{n+1} + C, & n \neq -1 \\ \ln|x| + C, & n = -1 \end{cases}$$

The Substitution Rule applies:

$$\int \frac{du}{u} = \ln|u| + C$$

Exercise:

Show that $\int \tan x\,dx = \ln|\sec x| + C$.

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{du}{u}$$

$u = \cos x \Rightarrow du = -\sin x\,dx$

$$= -\ln|u| + C = -\ln|\cos x| + C$$

$$= \ln|\sec x| + C \checkmark$$

The Chain Rule for log functions:

6.4b

$$\frac{d}{dx}\left[\ln(g(x))\right] = \frac{g'(x)}{g(x)} \rightarrow \frac{u'}{u}$$

Exercises:

Differentiate the function

$f(x) = \ln(\sin^2 x)$

slightly harder way

$f'(x) = \frac{2\sin x\cos x}{\sin^2 x}$

$f'(x) = 2\frac{\cos x}{\sin x}$

$f'(x) = 2\cot x$

slightly easier way

$f(x) = 2\ln(\sin x)$

$f'(x) = 2\frac{\cos x}{\sin x}$

$f'(x) = 2\cot x$

Ex: $f(x) = \ln(3x^2+5x) \Rightarrow f'(x) = \frac{6x+5}{3x^2+5x}$

$h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$

$$h'(x) = \frac{1 + \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2-1}}}{x+\sqrt{x^2-1}} \cdot \frac{\sqrt{x^2-1}}{\sqrt{x^2-1}}$$

$$h'(x) = \frac{\sqrt{x^2-1} + x}{\left(x+\sqrt{x^2-1}\right)\sqrt{x^2-1}} \Rightarrow h'(x) = \frac{1}{\sqrt{x^2-1}}$$

Don't rationalize denominators.

$\cancel{\frac{\sqrt{x^2-1}}{x^2-1}}$ $\frac{2}{\sqrt{3}}$ beautiful

Do simplify radicals:

$\sqrt{50} = 5\sqrt{2}$

$y = \ln(\csc x - \cot x)$

$y' = \frac{-\csc x \cot x + \csc^2 x}{\csc x - \cot x}$

$y' = \frac{\csc x(-\cot x + \csc x)}{\csc x - \cot x}$

$y' = \csc x$

Note: $\int \csc x \, dx = \ln|\csc x - \cot x| + C$

6.4c

$\int x \, dx = \frac{1}{2}x^2 + C$

Exercises:

$\int_0^2 x \, dx = 2$

Evaluate the integral.

$\frac{1}{5}\int_0^3 \frac{5\,dx}{5x+1}$

$u = 5x+1 \Rightarrow du = 5\,dx$

Limits

x	$u=5x+1$
3	16
0	1

$\frac{1}{5}\int_1^{16} \frac{1}{u}\,du$

$\frac{1}{5}\ln u \Big|_1^{16} = \frac{1}{5}\ln 16$

$\int \frac{\cos x}{2+\sin^2 x}\,dx$

$u = 2 + \sin x \Rightarrow du = \cos x \, dx$

$$\int \frac{du}{u} = \ln|u| + C$$
$$= \ln(2+\sin x) + C$$

6.4d

Calculus of logs and exponentials with bases other than *e*

since $\log_b x = \frac{\ln x}{\ln b}$ (change-of-base formula), where b is a constant, so ln b is a constant, we have

$$\frac{d}{dx}(\log_b x) = \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right) = \frac{1}{\ln b \, x}$$

So $\boxed{\frac{d}{dx}[\log_b x] = \frac{1}{x \ln b}}$

since $b = e^{\ln b}$, $b^x = (e^{\ln b})^x = e^{x \ln b}$,

$$\frac{d}{dx}(b^x) = \frac{d}{dx}(e^{x\ln b}) = \ln b \, e^{x \ln b}$$
$$= \ln b \cdot b^x$$

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$$

$\frac{d}{dx}[\ln x] = \frac{1}{x}$ $\quad$ $\frac{d}{dx}[\ln(u(x))] = \frac{u'(x)}{u(x)}$ Chain Rule

$\int \frac{1}{x} dx = \ln|x| + C$ $\quad$ $\int \frac{du}{u} = \ln|u| + C$

$\frac{d}{dx}(e^x) = e^x$ $\quad \int e^x\,dx = e^x + C$ $\quad \frac{d}{dx}\left[e^{u(x)}\right] = u'(x)e^{u(x)}$

$$\frac{d}{dx}(b^x) = \ln b \cdot b^x$$

$$\int b^x\,dx = \frac{b^x}{\ln b} + C$$

From 6.2, $f(x) = b^x$

$\Rightarrow f'(x) = b^x \lim\limits_{x\to 0} \frac{b^x - 1}{x}$,

so $\lim\limits_{x\to 0} \frac{b^x-1}{x} = \ln b$

6.4e

Definition: (Euler's number *e*)

For $\int \frac{du}{u}$, consider $\int \frac{\cos x}{\sin x}dx$, $\int \frac{2x+1}{x^2+x}dx$

$\int \frac{e^x}{e^x+1}dx$

Recall for $f(x) = \ln x$, $f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$

But $f'(1) = \lim\limits_{\cancel{h}\,x \to 0} \frac{f(1+\cancel{h}\,x) - f(1)}{\cancel{h}\,x}$

$= \lim\limits_{x\to 0} \frac{\ln(1+x) - \ln(1)}{x}$ $\quad \ln(1) \to 0$

$f'(1) = \lim\limits_{x\to 0} \frac{1}{x}\ln(1+x) = \lim\limits_{x\to 0} \ln(1+x)^{1/x}$

So $1 = \lim\limits_{x\to 0} \ln(1+x)^{1/x} \Rightarrow e = \lim\limits_{x\to 0}(1+x)^{1/x}$

Replace $n = \frac{1}{x} \rightarrow e = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

$$e = \lim_{x\to 0}(1+x)^{\frac{1}{x}}$$

limit definitions of e

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

6.4f

Exercise:

Use logarithmic differentiation to find the derivative of the function.

$$y = \frac{e^{-x}\cos^2 x}{x^2+x+1}$$

$$\ln y = \ln\left(\frac{e^{-x}\cos^2 x}{x^2+x+1}\right)$$

$$\ln y = \cancel{\ln}\, \cancel{e}^{-x} + 2\ln \cos x - \ln(x^2+x+1)$$

$$\ln y = -x + 2\ln \cos x - \ln(x^2+x+1)$$

$$\frac{y'}{y} = -1 + 2\frac{(-\sin x)}{\cos x} - \frac{2x+1}{x^2+x+1}$$

$$y' = y\left(-1 - 2\tan x - \frac{2x+1}{x^2+x+1}\right)$$

$$\boxed{y' = \frac{e^{-x}\cos^2 x}{x^2+x+1}\left(-1 - 2\tan x - \frac{2x+1}{x^2+x+1}\right)}$$

6.4g

Exercise:

Use logarithmic differentiation to find the derivative of the function.

$$y = x^{\cos x}$$

$\ln y = \ln x^{\cos x}$ $\quad u = \cos x \quad v = \ln x$

$\ln y = \cos x \ln x \quad u' = -\sin x \quad v' = \frac{1}{x}$

$\frac{y'}{y} = -\sin x \ln x + \frac{1}{x}\cos x$

$y' = y\left(\frac{1}{x}\cos x - \sin x \ln x\right)$

$y' = x^{\cos x}\left(\frac{1}{x}\cos x - \sin x \ln x\right)$

Process for logarithmic differentiation

Take the natural log of both sides

Use properties of logs

Differentiate (usually with respect to x)—use implicit differentiation for y

Give $\frac{dy}{dx}$ or y' in terms of x (or whatever the independent variable is)

Summary of techniques involving exponents

(variable base)$^{\text{constant exponent}}$ –power rule (ex: $y = x^2$)

(constant base)$^{\text{variable exponent}}$ –rules for exponentials (ex: $y = e^x$ or $y = 2^x$)

(variable base)$^{\text{variable exponent}}$ –logarithmic differentiation

$y = x^{\cos x} \qquad y = x^x$

$\frac{d}{dx}(\cos x) = -\sin x$, $\frac{d}{dx}(\sin x) = \cos x$

$\int \sin x\, dx = -\cos x + C$ $\int \cos x\, dx = \sin x + C$

$\sin^2\theta + \cos^2\theta = 1$

$\frac{d}{dx}(\sec x) = \sec x \tan x$ $\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\csc x) = -\csc x \cot x$

$\int \tan x\, dx = \ln|\sec x| + C$

$\tan^2\theta + 1 = \sec^2\theta$

More practice:

Differentiate

1. $g(t) = \sqrt{1 + \ln t}$

2. $F(t) = 3^{\cos 2t}$

3. $y = \sqrt{x}\, e^{x^2 - x}(x+1)^{2/3}$

4. $y = (\sin x)^{\ln x}$

5. $g(x) = x\sin(2^x)$

1. $g(t) = \sqrt{1+\ln t} \Rightarrow g'(t) = \dfrac{1/t}{2\sqrt{1+\ln t}} = \dfrac{1}{2t\sqrt{1+\ln t}}$

2. $F(t) = 3^{\cos 2t} \Rightarrow F'(t) = \ln 3 \cdot 3^{\cos 2t}(-2\sin 2t)$

$$= -2\ln 3 \cdot 3^{\cos 2t}\sin 2t$$

3. $y = \sqrt{x}\, e^{x^2-x}(x+1)^{2/3}$

$$\ln y = \tfrac{1}{2}\ln x + x^2 - x + \tfrac{2}{3}\ln(x+1)$$

$$\frac{y'}{y} = \frac{1}{2x} + 2x - 1 + \frac{2}{3(x+1)}$$

$$y' = \sqrt{x}\, e^{x^2-x}(x+1)^{2/3}\left[\frac{1}{2x} + 2x - 1 + \frac{2}{3(x+1)}\right]$$

4. $y = (\sin x)^{\ln x} \Rightarrow \ln y = \ln(\sin x)^{\ln x}$

$$\ln y = \ln x \cdot \ln(\sin x)$$

$u = \ln x \quad v = \ln(\sin x)$

$u' = \frac{1}{x} \quad v' = \cot x$

$$\frac{y'}{y} = \frac{1}{x}\ln(\sin x) + \ln x \cot x$$

$$y' = (\sin x)^{\ln x}\left(\frac{\ln(\sin x)}{x} + \ln x \cot x\right)$$

5. $g(x) = \underbrace{x}\,\underbrace{\sin(\underbrace{2^x})}$ $\qquad u = \underbrace{x} \qquad v = \underbrace{\sin(\underbrace{2^x})}$

$$u' = 1 \qquad v' = \ln 2 \cdot 2^x \cos(2^x)$$

$$g'(x) = \sin(2^x) + x \ln 2 \cdot 2^x \cos(2^x)$$

$$f(x) = x \cdot x^{e^x} \qquad f'(x) = xe^x + x$$

$$u = x \qquad \boxed{v = x^{e^x}}$$

$$u' = 1 \qquad \ln v = e^x \ln x$$

$$\frac{v'}{v} = e^x \ln x + \frac{e^x}{x}$$

$$v' = x^{e^x}\left(e^x \ln x + \frac{e^x}{x}\right)$$