6.3 - Logarithmic Functions

6.3a

Graphs and limits of logarithmic functions

For $f(x) = b^x$ $(b>0, b \neq 1)$, $f^{-1}(x) = \log_b x$.

x	3^x
-2	$1/9$
-1	$1/3$
0	1
1	3
2	9
3	27

3^x	x
$1/9$	-2
$1/3$	-1
1	0
3	1
9	2
27	3

logarithmic form

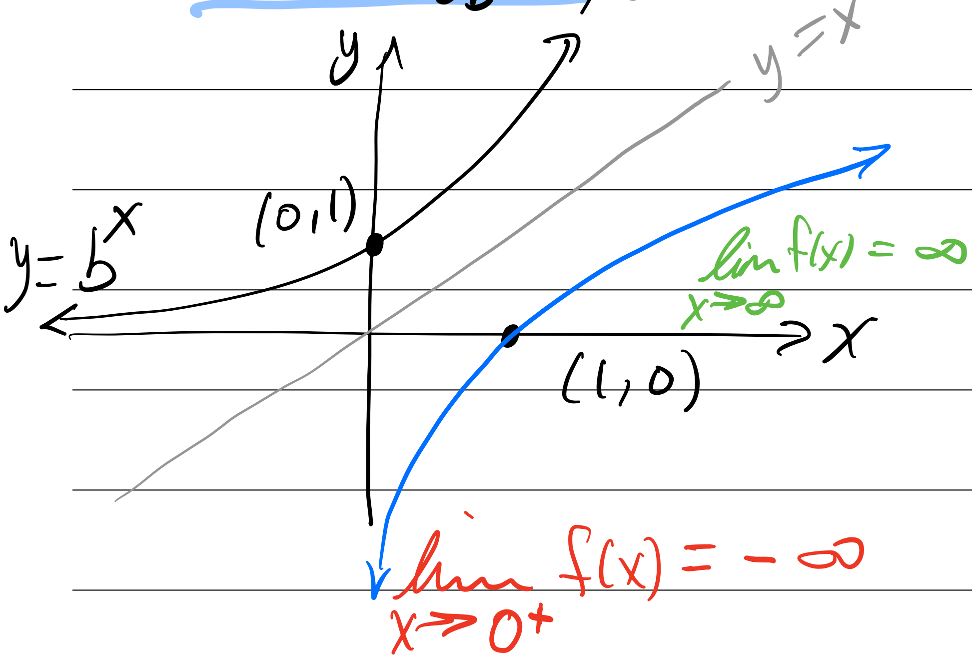
$\boxed{\log_3 9 = 2}$, because

$\boxed{3^2 = 9}$ exponential form

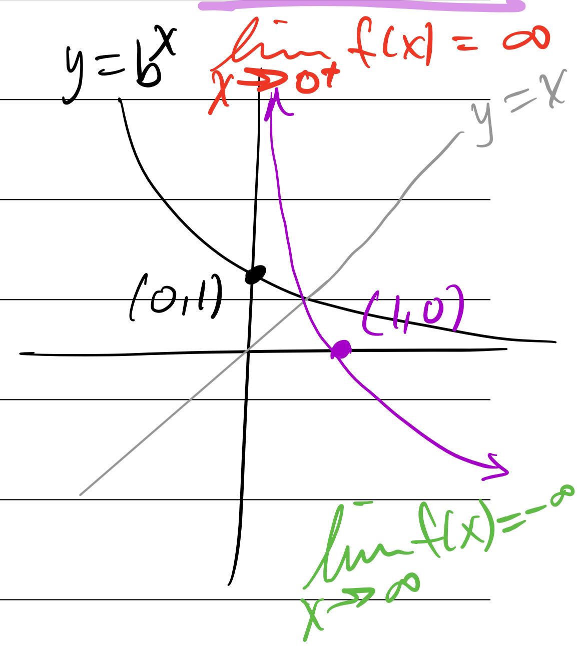
A logarithm is an exponent

$\log_5 125$ - What is the exponent on 5 that gives 125?

$f(x) = \log_b x$, $b>1$



$0 < b < 1$



Dom: $(0, \infty)$

6.3b

Properties of logarithms

① $\log_b b^x = x$
② $b^{\log_b x} = x$

cancellation properties

③ $\log_b MN = \log_b M + \log_b N$

④ $\log_b \frac{M}{N} = \log_b M - \log_b N$

⑤ $\log_b M^r = r \log_b M$

If $b = e \approx 2.718$, then $\log_e x = \ln x$.

If $b = 10$, then $\log_{10} x = \log x$

Pf of ③: Let $x = \log_b M$ and $y = \log_b N$.
Then $M = b^x$, $N = b^y$.

$$\log_b MN = \log_b b^x b^y = \log_b b^{x+y}$$

$$= x + y = \log_b M + \log_b N \checkmark$$

Consider $\log_b M^3 = \log_b (MMM) = \log_b M + \log_b M + \log_b M$

$$= 3 \log_b M$$

Change-of-base formula

$$\log_b x = \frac{\log_a x}{\log_a b} = \frac{\ln x}{\ln b} \text{ (using } a = e)$$

Justification: Let $\log_b x = y$.
Then $b^y = x$

$$\Rightarrow \quad \log_a b^{y} = \log_a x$$

$$y \log_a b = \log_a x$$

$$\log_b x = y = \frac{\log_a x}{\log_a b} \quad \checkmark$$

6.3c

Exercises:

Find the limit

$\lim\limits_{x \to 2^-} \log_5 \left(8x - x^4\right)$

$$\lim_{x \to 2^-} \log_5 \left[x(8 - x^3)\right] = \lim_{x \to 2^-} \left[\log_5 x + \log_5 (8 - x^3)\right]$$

$$= \log_5 2 + \lim_{x \to 2^-} \log_5 (8 - x^3) = -\infty$$

As $x \to 2^-$, $8 - x^3 \to 0^+$

$\lim\limits_{x \to \infty} [\ln(2 + x) - \ln(1 + x)]$

$$\lim_{x \to \infty} \ln \frac{2+x}{1+x} = \ln \boxed{\lim_{x \to \infty} \frac{2+x}{1+x} \frac{\frac{1}{x}}{\frac{1}{x}}}$$

$$\ln \lim_{x \to \infty} \frac{\cancel{\frac{2}{x}}^{0} + 1}{\cancel{\frac{1}{x}}^{0} + 1} = \ln 1 = 0$$

6.3d

$\frac{d}{dx}\left(e^{u(x)}\right) = u'(x)e^{u(x)}$

Exercise:

On what interval is the curve $y = 2e^x - e^{-3x}$ concave downward?

$y'' < 0$

• Find y'', • solve $y'' = 0$, • solve $y'' < 0$

$y' = 2e^x + 3e^{-3x}$

$y'' = 2e^x - 9e^{-3x}$ → $2e^x - \frac{9}{e^{3x}} = 0$

$y'' = 0 \Rightarrow e^{3x}\left(2e^x - 9e^{-3x}\right) = 0 \cdot e^{3x}$

$2e^{4x} - 9 = 0 \Rightarrow \cancel{\ln}\left(\cancel{e}^{4x}\right) = \ln\left(\frac{9}{2}\right)$

$4x = \ln\frac{9}{2} \Rightarrow x = \frac{1}{4}\ln\frac{9}{2}$

y'' neg | pos

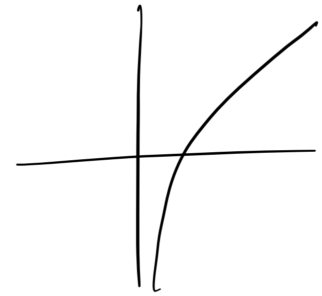
$x = 0$ $\quad \frac{1}{4}\ln\frac{9}{2} \approx 0.376$ $\quad x = 2$

The curve is concave down on $\left(-\infty, \frac{1}{4}\ln\frac{9}{2}\right)$.

For $\log_b \square$, $\square > 0$.

Ex: Find the domain of

$$f(x) = \log_2 (x^2 - 3x)$$

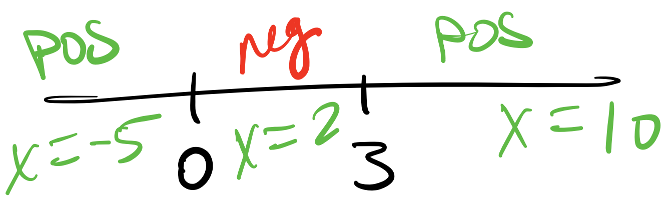


$y = \log_2 x$

$x > 0$

$$x^2 - 3x > 0$$

$$x(x-3) > 0$$



Dom: $(-\infty, 0) \cup (3, \infty$

Ex. Domain of

$f(x) = \log(x^2+1)$ is $\mathbb{R}$.