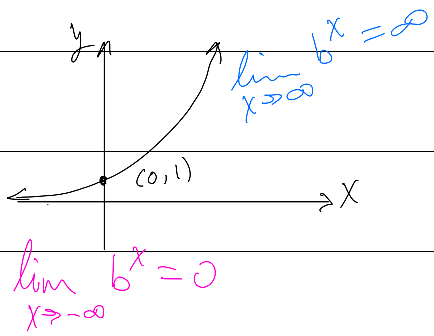
6.2 - Exponential Functions and Their Derivatives

6.2a

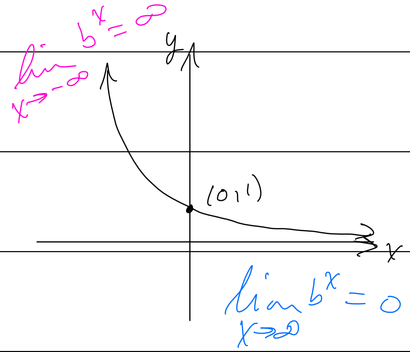
The derivative of an exponential function using the definition of a derivative

Consider $f(x) = b^x$, $b > 0$, $b \neq 1$, $x \in \mathbb{R}$

$b > 1$



$0 < b < 1$



$$f'(x) = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} = \lim_{h \to 0} \frac{b^x(b^h - 1)}{h}$$

$$= b^x \underbrace{\boxed{\lim_{h \to 0} \frac{b^h - 1}{h}}}_{\text{constant}}$$

we will revisit this limit in 6.4.

The derivative is a constant multiple of the original function (if the limit exists).

The derivative of the natural exponential function

Consider $b = 2$ & $b = 3$.

$$\lim_{h \to 0} \frac{2^h - 1}{h} \doteq 0.69 < 1$$

h	$\frac{2^h - 1}{h}$
0.01	0.695555
0.001	0.693387
0.0001	0.693150
10^{-5}	0.693147

$$\lim_{h\to 0} \frac{3^h-1}{h} \approx 1.1 > 1$$

It is reasonable to expect that there is

$b \in (2,3) \ni \lim_{h\to 0} \frac{b^h-1}{h} = 1.$

For $b = e \approx 2.718$, $\lim_{h\to 0} \frac{e^h-1}{h} = 1$

by def of e.

So if $f(x)=e^x$, then $f'(0)=1$,

and $\frac{d}{dx}(e^x) = e^x$.

Note:

Since $b^0=1$, the limit can be rewritten $\lim_{h\to 0} \frac{b^h-b^0}{h-0}$

So $f'(x) = f'(0)\, f(x)$

$y = \cos(x^3)$ $\qquad$ $y' = \cancel{-\sin(3x^2)}$

Exercise:

Find y' and y'' for ~~$y = e^{x^2}$~~. $y = e^{x^2}$

$$y' = 2x\, e^{x^2} \qquad u = 2x \quad v = e^{x^2}$$

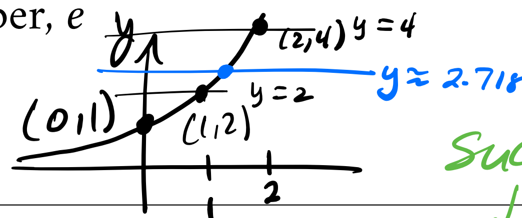
$$u' = 2 \quad v' = 2x e^{x^2}$$

$$y'' = 2e^{x^2} + 4x^2 e^{x^2}$$

constant

Recall $\dfrac{d}{dx}[b^x] = b^x \boxed{\lim_{h \to 0} \dfrac{b^h - 1}{h}} = k\, b^x$

A derivation of the value of Euler's number, e



Consider $f(x) = 2^x$

such that

Because it's continuous, there is $c \ni 2^c = e$.

Then $e^x = (2^c)^x = 2^{cx}$

Differentiate: $e^x = k c 2^{cx}$ Let $x = 0$:

$$1 = kc \Rightarrow c = 1/k$$

$$k \approx 0.693147 = \lim_{h \to 0} \frac{2^h - 1}{h} \Rightarrow e = 2^{1/k} \approx 2.718\ldots$$

The integral of the natural exponential function

Since $\boxed{\dfrac{d}{dx}(e^x) = e^x}$,

$$\boxed{\int e^x\,dx = e^x + C}$$

6.2b

Exercises:

Differentiate the function.

$$y = \frac{e^x}{1 - e^x}$$

$u = e^x \qquad v = 1 - e^x \qquad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$u' = e^x \qquad v' = -e^x$

$$y' = \frac{e^x(1-e^x) + e^{2x}}{(1-e^x)^2}$$

$$y' = \frac{e^x}{(1-e^x)^2}$$

Evaluate the integral.

$$\frac{1}{3}\int 3x^2 e^{x^3}\, dx$$

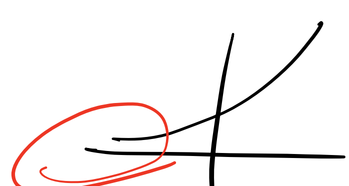
Let $u = x^3$. Then $du = 3x^2 dx$

$$\frac{1}{3}\int e^u\, du = \frac{1}{3}e^u + C$$

$$= \frac{1}{3}e^{x^3} + C$$

Find the limit.

$\lim_{x\to-\infty} (1.001)^x$ ~ $1.001 > 1$



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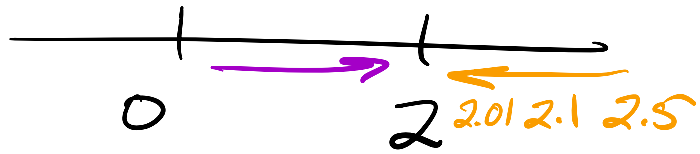
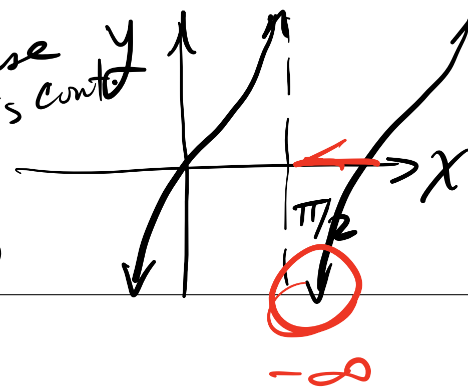


$x^{-3} = \frac{1}{x^3}$ $x \to 2^-$ $x \to 2^+$

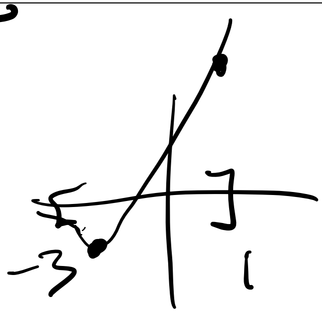
$$\lim_{x\to(\pi/2)^+} e^{\tan x} = e^{\lim_{x\to(\pi/2)^+} \tan x}$$ because e^x is cont.



as $x \to (\pi/2)^+$, $\tan x \to -\infty$

$e^{-\text{large}} = \frac{1}{e^{\text{large}}}$ 0

6.2c

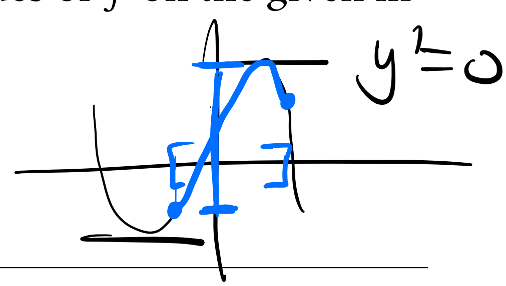


Exercise:

Find the absolute maximum and absolute minimum values of f on the given interval.

$f(x) = xe^{\frac{x}{2}}$, $[-3,\ 1]$

local max
local min



$f(x) = x\,e^{x/2}$

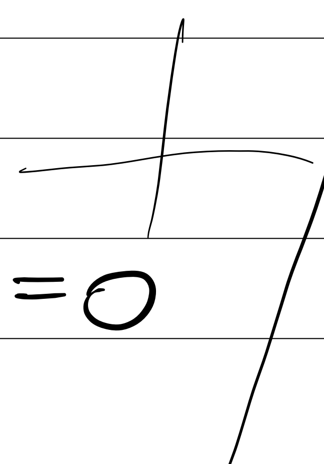
$u = x$ $v = e^{\frac{1}{2}x}$

$u' = 1$ $v' = \frac{1}{2} e^{x/2}$

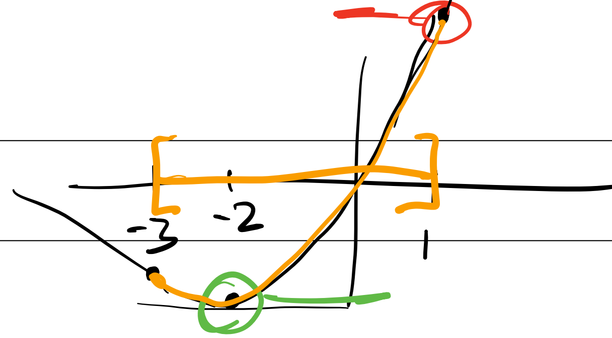
$f'(x) = e^{x/2} + \frac{1}{2} x e^{x/2}$

$f'(x) = 0 \Rightarrow (1 + \frac{1}{2}x) e^{x/2} = 0$

$1 + \frac{1}{2}x = 0$



$x = -2$



endpoints

	x	$y = xe^{x/2}$
	-3	≈ -0.67
c.n.	-2	$\approx -0.74 \leftarrow$ abs min
	1	$\approx 1.65 \leftarrow$ abs max

$y = 3x + 2$

x	y
2	8
3	11
4	14