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6.1 - Inverse Functions

6.1a

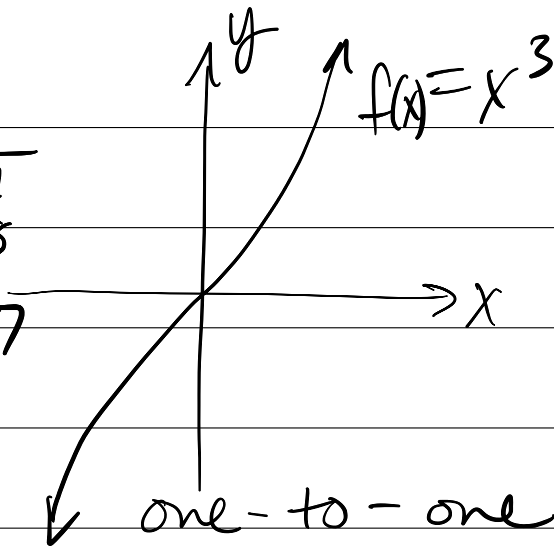
Definition: (one-to-one)

A function f is one-to-one if $f(x_2) \neq f(x_1)$ whenever $x_2 \neq x_1$.

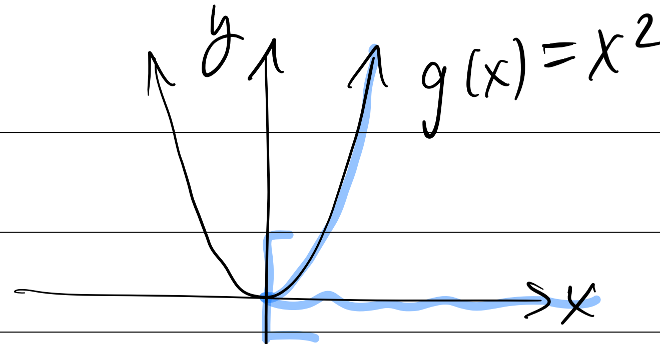
Description of inverse functions

$y=x^2, \ x \geq 0$

x	y
1	1
2	8
3	27



one-to-one



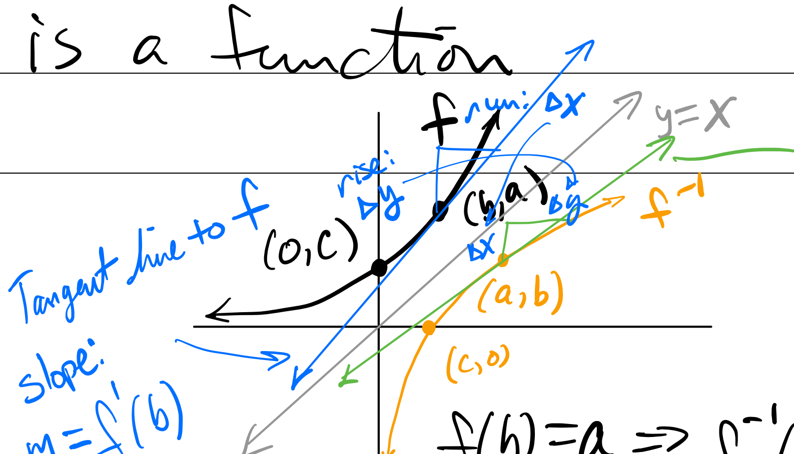
If we restrict the domain, $[0,\infty)$, we have a piece that is one-to-one.

not one-to-one

The inverse of a function reverses inputs and outputs

x's y's

The inverse of a one-to-one function is a function



slope: $m=(f^{-1})'(a)=\dfrac{1}{f'(b)}$

$f(b)=a \Rightarrow f^{-1}(a)=b \quad (f^{-1})'(a)=\dfrac{1}{f'(f^{-1}(a))}$

6.1b

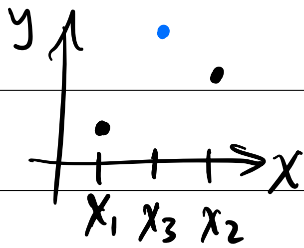
Theorems (statement and proof):

If f is one-to-one and continuous on $[a,\ b]$, then f^{-1} is also continuous.

mostly a

pf: f is one-to-one $\Rightarrow$ f is increasing or decreasing (can't change direction)

Suppose there exist x_1, x_2, x_3 such that $x_1 < x_3 < x_2$.

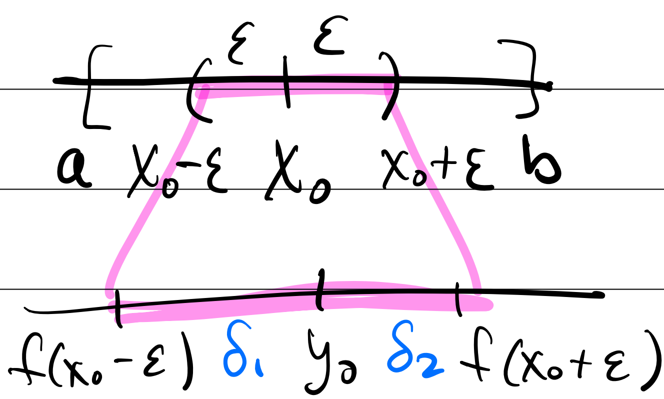


Suppose $f(x_2) > f(x_1)$

Suppose $f(x_3) > f(x_2)$

Because f is cont., by the IVT, there is $c \in (x_1, x_3)$ such that $f(c) = f(x_2)$, which contradicts that f is one-to-one. (similarly if $f(x_3) < f(x_1)$).

So f is increasing. Then for every $\varepsilon > 0$ such that $(x_0 - \varepsilon, x_0 + \varepsilon) \in (a, b)$,



f maps $(x_0 - \varepsilon, x_0 + \varepsilon)$ into $(f(x_0 - \varepsilon), f(x_0 + \varepsilon))$

Let $f(x_0) = y_0$ Let $\delta = \min\{\delta_1, \delta_2\}$.

Then $(y_0 - \delta, y_0 + \delta) \in (f(x_0 - \varepsilon), f(x_0 + \varepsilon))$

That is, for $\varepsilon > 0$, there is δ such that if $|y - y_0| < \delta$, then $|f^{-1}(y) - f^{-1}(y_0)| < \varepsilon$.

$$\Rightarrow \lim_{y \to y_0} f^{-1}(y) = f^{-1}(y_0) = x_0$$

$\Rightarrow f^{-1}$ is continuous.

Suppose that f is one-to-one and differentiable with inverse function f^{-1}.

Cont. Consider $(f^{-1})'(a)$ (Note: a is a y-value for f)

$$(f^{-1})'(a) = \lim_{x \to a} = \frac{f^{-1}(x) - f^{-1}(a)}{x - a}$$

If we let $f^{-1}(a) = b$, $f^{-1}(x) = y$, $f(b) = a$, $f(y) = x$

then $(f^{-1})'(a) = \lim_{x \to a} \dfrac{y - b}{f(y) - f(b)} = \boxed{\lim_{y \to b} \dfrac{y - b}{f(y) - f(b)}}$

f cont $\Rightarrow f^{-1}$ cont. $\Rightarrow \lim_{x \to a} f^{-1}(x) = f^{-1}(a)$

$\Rightarrow \lim_{x \to a} y = b$ so as $x \to a$, $y \to b$

f is continuous at a if $\lim_{x\to a} f(x) = f(a)$.

$$\Rightarrow (f^{-1})'(a) = \frac{1}{\lim_{y\to b}\frac{f(y)-f(b)}{y-b}} = \frac{1}{\lim_{y\to b}\frac{f(y)-f^{-1}(a)}{y-f^{-1}(a)}}$$

$$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}$$

This generalizes if we replace a with x.

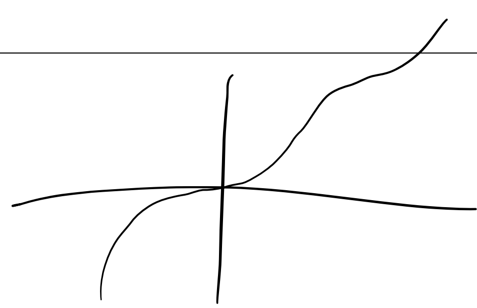
Alternative approach to developing this formula: (Implicit differentiation)

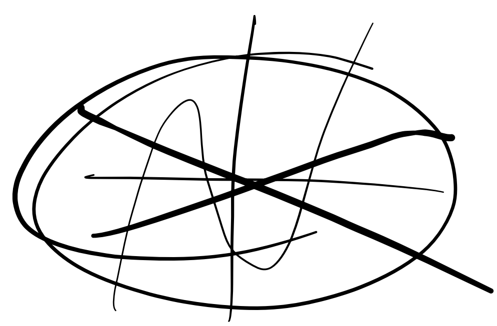
If $y = f^{-1}(x)$, then $f(y) = x$

so $f'(y)\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{f'(y)}$

chain Rule

$$(f^{-1})'(x) = \frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))}$$





6.1c

Exercises:

If $f(x) = x^5 + x^3 + x$, find $f^{-1}(3)$ and $f\left(f^{-1}(2)\right)$.

↑ The x-value that gives $y = 3$

$f^{-1}(3)$: $3 = x^5 + x^3 + x \Rightarrow x = 1$

$f^{-1}(3) = 1$

$f(f^{-1}(2)) = 2$

If f is one-to-one

Is f one-to-one?

$f'(x) = 5x^4 + 3x^2 + 1 > 0$ always increasing

$\Rightarrow f$ is one-to-one.

Find $\left(f^{-1}\right)'(a)$ for $f(x) = x^3 + 3\sin x + 2\cos x$ and $a = 2$.

$$\left(f^{-1}\right)'(a) = \frac{1}{\boxed{f'(\boxed{f^{-1}(a)})}}$$

Need:

- $f' = 3x^2 + 3\cos x - 2\sin x$
- $f^{-1}(2)$ (x-value that gives $y = 2$) $\Rightarrow 2 = x^3 + 3\sin x + 2\cos x$

$x = 0$ so $f(0) = 2 \Rightarrow f^{-1}(2) = 0$

$f'(0) = 3 \Rightarrow \left(f^{-1}\right)'(2) = \frac{1}{3}$

6.1d

f is one-to-one if

if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Exercise:

Let $f(x) = \sqrt{x-2}$ and $a = 2$.

a) Show that f is one-to-one

Option 1:

Suppose $f(x_1) = f(x_2)$

$\Rightarrow \sqrt{x_1 - 2} = \sqrt{x_2 - 2} \Rightarrow x_1 - 2 = x_2 - 2$

$x_1 = x_2$.

Option 2:

$f'(x) = \frac{1}{2\sqrt{x-2}} > 0$

inc on $(2, \infty)$

Aside: Consider $g(x)=x^2$.

$$g(x_1)=g(x_2) \Rightarrow x_1^2 = x_2^2$$
$$x_1 = \pm x_2$$

Find $\left(f^{-1}\right)'(a) = \dfrac{1}{\boxed{f'\left(f^{-1}(a)\right)}}$ $\quad f(x)=\sqrt{x-2}=(x-2)^{1/2}, \; a=2$

Need $\cdot\; f^{-1}(2)$: $2=\sqrt{x-2} \Rightarrow 4 = x-2$

(need x-value) $\quad x=6=f^{-1}(2)$

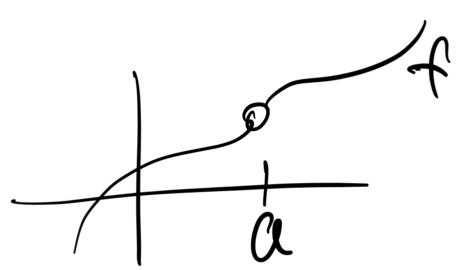
$\cdot\; f'(x) = \dfrac{1}{2\sqrt{x-2}}$

$f'(6) = \dfrac{1}{4} \Rightarrow \left(f^{-1}\right)'(2) = 4$

✓ conditional

If p, then q $\qquad p \Rightarrow q$

If f is diff'able at a, then f is cont. at a.



f is not defined at a.

f is not cont. at a.

f is not diff'able at a

↳ If f is not cont. at a, then f is not diff'able at a.

If not q, then not p.

This is the contrapositive of the conditional