

11.1 - Sequences

11.1a

A sequence is a set of numbers in a particular order. OR a function whose domain is $0, 1, 2, 3, \dots$ OR $1, 2, 3, \dots$

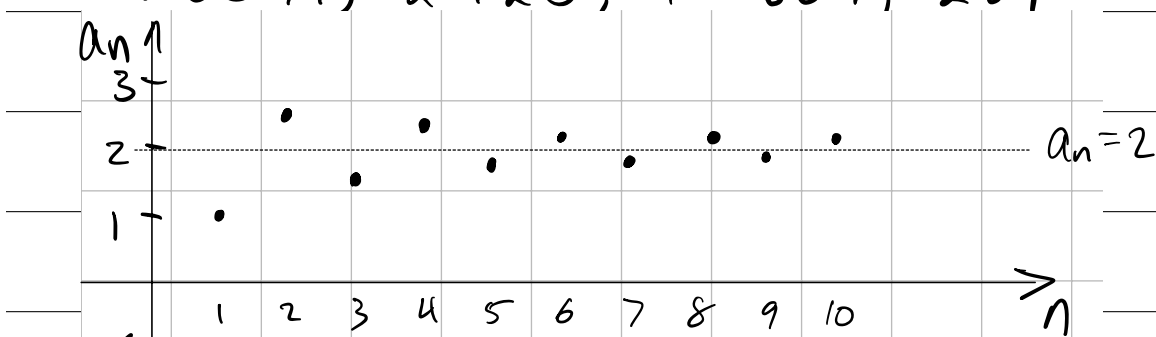
Exercise:

Calculate, to four decimal places, the first ten terms of the sequence and use them to plot the graph of the sequence by hand. Does the sequence appear to have a limit? If so, calculate it. If not, explain why.

$$a_n = 2 + \frac{(-1)^n}{n}$$

n^{th} term of the sequence

$$\{a_n\}_{n=1}^{10} \approx 1, 2.5, 1.6667, 2.25, 1.8, 2.1667, 1.8571, 2.125, 1.8889, 2.1$$



This sequence is convergent.

The limit appears to be 2.

$$\lim_{n \rightarrow \infty} \left[2 + \frac{(-1)^n}{n} \right] = \lim_{n \rightarrow \infty} 2 + \lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 2$$

Theorem (statement):

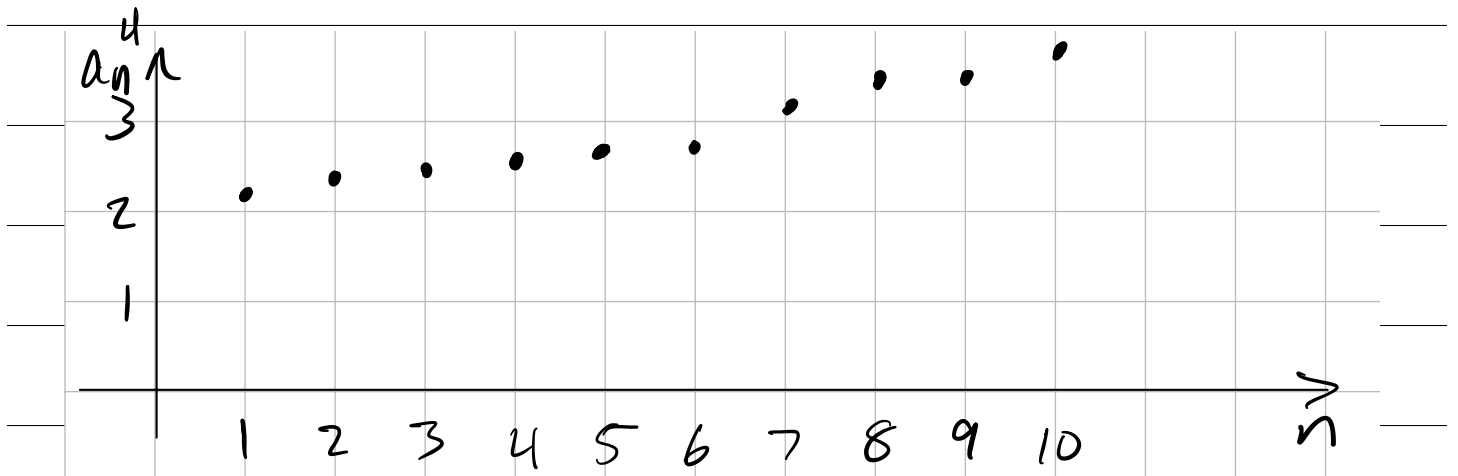
If $\lim_{n \rightarrow \infty} |a_n| = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Definition: A sequence $\{a_n\}$ has the **limit** L and we write $\lim_{n \rightarrow \infty} a_n = L$ or $a_n \rightarrow L$ as $n \rightarrow \infty$ if for every $\varepsilon > 0$ there is a corresponding integer N such that if $n > N$, then $|a_n - L| < \varepsilon$.

Exercise: $a_1, a_2, a_3, \dots$

$$a_n = 1 + \frac{10^n}{9^n}$$

$$\{a_n\}_{n=1}^{\infty} \approx 2.1111, 2.2346, 2.3717, 2.5242, 2.6935, 2.8817, 3.0908, 3.3231, 3.5812, 3.8680$$



this does not appear to have a limit

$\lim_{n \rightarrow \infty} \left[1 + \left(\frac{10}{9} \right)^n \right] = \infty$. The sequence is divergent.

11.1b

Exercises:

Determine whether the sequence converges or diverges. If it converges, find the limit.

$$a_n = \frac{3+5n^2}{1+n} \quad \frac{3+5n^2}{1+n} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{\frac{3}{n} + 5n}{\frac{1}{n} + 1} = \infty$$

The sequence diverges.

$$a_n = \frac{4^n}{1+9^n} \quad \frac{4^n}{1+9^n} \cdot \frac{\frac{1}{9^n}}{\frac{1}{9^n}} = \lim_{n \rightarrow \infty} \frac{(\frac{4}{9})^n}{(\frac{1}{9})^n + 1} = 0$$

Note: $\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$ (Convergent)
and divergent otherwise

Alternative: $0 \leq \frac{4^n}{1+9^n} \leq \frac{4^n}{9^n}$

↑ bigger denom $\Rightarrow$ smaller fraction

$$0 = \lim_{n \rightarrow \infty} 0 \leq \lim_{n \rightarrow \infty} \frac{4^n}{1+9^n} \leq \lim_{n \rightarrow \infty} \frac{4^n}{9^n} = 0$$

$\lim_{n \rightarrow \infty} \frac{4^n}{1+9^n} = 0$ by the Squeeze Theorem.
(Convergent)

$$a_n = e^{\frac{2n}{n+2}}$$

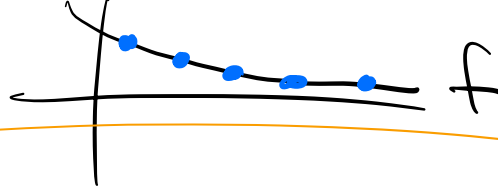
Exponent: $\lim_{n \rightarrow \infty} \frac{2n}{n+2} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{2}{1+\frac{2}{n}} = 2$

side note $\lim_{n \rightarrow \infty} \frac{2n}{n+2} = 2$

$$\lim_{n \rightarrow \infty} e^{\frac{2n}{n+2}} = e^{\lim_{n \rightarrow \infty} \frac{2n}{n+2}} = e^2 \quad (\text{Convergent})$$

OR l'Hospital's Rule: $\lim_{x \rightarrow \infty} \frac{2x}{x+1} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{2}{1} = 2$

Theorems (statement):



If $\lim_{x \rightarrow \infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n \rightarrow \infty} a_n = L$.

If $\lim_{n \rightarrow \infty} a_n = L$ and the function f is continuous at L , then $\lim_{n \rightarrow \infty} f(a_n) = f(L)$.

we used this when we used l'Hospital's Rule

11.1c

$$\frac{d}{dx} (\ln 2x) = \frac{1}{2x} = \frac{1}{x}$$

Exercises:

we used this in the previous exercise 1st time

Determine whether the sequence converges or diverges. If it converges, find the limit.

$$\left\{ \frac{\ln n}{\ln(2n)} \right\} \quad a_n = \frac{\ln n}{\ln(2n)} \rightarrow \text{use } f(x) = \frac{\ln x}{\ln(2x)}$$

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\ln(2x)} = \lim_{x \rightarrow \infty} \frac{\ln x}{\ln 2 + \ln x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1/x}{1/x} = 1$$

$$\text{So } \lim_{n \rightarrow \infty} \frac{\ln n}{\ln(2n)} = 1. \quad (\text{Convergent})$$

$$a_n = \frac{(-3)^n}{n!}$$

Recall: $4! = 4 \cdot 3 \cdot 2 \cdot 1$

Def: $n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$

$$0! = 1$$

$$a_n = \frac{(-3)(-3)(-3) \cdots (-3)}{1 \cdot 2 \cdot 3 \cdots n}$$

$$\text{If } n > 3, \left| \frac{-3}{n} \right| < 1$$

$$0 \leq |a_n| = \frac{3^n}{n!}$$

less than 1

$$= \frac{3}{1} \cdot \frac{3}{2} \cdot \frac{3}{3} \cdots \frac{3}{n-1} \cdot \frac{3}{n}$$

$$\leq \frac{3}{1} \cdot \frac{3}{2} \cdot \frac{3}{3} \cdots \frac{3}{n}$$

$$\text{So } 0 \leq |a_n| \leq \frac{27}{2n}$$

$$0 = \lim_{n \rightarrow \infty} 0 \leq \lim_{n \rightarrow \infty} |a_n| \leq \lim_{n \rightarrow \infty} \frac{27}{2n} = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} |a_n| = 0 \text{ (Squeeze Theorem)}$$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n = 0 \text{ (convergent)}$$

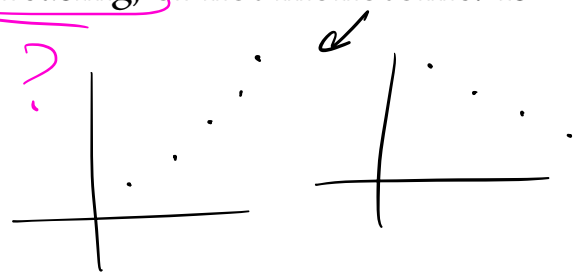
11.1d

always increasing or
always decreasing

Determine whether the sequence is increasing, decreasing, or not monotonic. Is the sequence bounded?

$$a_n = \frac{1-n}{2+n}$$

$$a_n = a(n) \quad ?$$



$$\text{Explore: } a_1 = 0$$

$$a_2 = -\frac{1}{4} \quad -0.25$$

$$a_3 = -\frac{2}{5} \quad -0.4$$

$$a_4 = -\frac{3}{6} = -\frac{1}{2} \quad -0.5$$

$$a_5 = -\frac{4}{7}$$

smaller than $-\frac{1}{2}$

We suspect this is decreasing.

that is, we think $a_{n+1} < a_n$.

$$\frac{-n}{n+3} = \frac{1-(n+1)}{2+(n+1)} < \frac{1-n}{2+n}$$

$$n+3 > 0, n+2 > 0$$

$$\text{clear the fractions } \Leftrightarrow -n(2+n) < (1-n)(n+3)$$

$$\Leftrightarrow -n^2 - 2n < -n^2 - 2n + 3$$

$$\Leftrightarrow 0 < 3$$

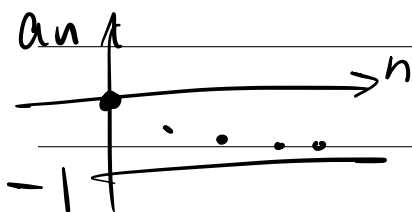
Alternative approach: Let $f(x) = \frac{1-x}{2+x}$

$$f'(x) = \frac{-(2+x) - (1-x)}{(2+x)^2} = \frac{-3}{\text{positive}} < 0$$

The sequence is decreasing.

Bounded? $a_n \leq 0$. $\lim_{n \rightarrow \infty} \frac{1-n}{2+n} \cdot \frac{1}{n} = \lim_{n \rightarrow \infty} \frac{1/n - 1}{2/n + 1}$

$$\lim_{n \rightarrow \infty} a_n = -1$$



$\{a_n\}$ is bounded.

Theorem (statement):

Monotonic Sequence Theorem: Every bounded, monotonic sequence is convergent.

11.1e

Exercise:

Show that the sequence defined by

$$a_1 = 1, a_{n+1} = 3 - \frac{1}{a_n}$$

$$a_2 = 3 - \frac{1}{1} = 2$$

$$a_3 = 3 - \frac{1}{2} = 2.5$$

satisfies $0 < \underline{a_n} < 2$ and is decreasing. Deduce that the sequence is convergent and find its limit.

We will use mathematical induction.

$$a_n < 3$$

Step 1: $n=1: a_1 = 1 < 3 \checkmark$

Step 2: Assume $a_k < 3$ for some k .
 $\Leftrightarrow -a_k > -3$

$$\Leftrightarrow -\frac{1}{a_k} < -\frac{1}{3}$$

$$\Leftrightarrow 3 - \frac{1}{a_k} < 3 - \frac{1}{3} < 3$$

Step 3:

$$\Leftrightarrow a_{k+1} < 3$$

$a_n < 3$ for all $n > 1$.

increasing ($a_{n+1} > a_n$)

Step 1: $a_2 > a_1$ $a_2 = 2, a_1 = 1$

so $a_2 > a_1$ ✓

Step 2: Assume $a_{k+1} > a_k$

Step 3: $\Leftrightarrow -a_{k+1} < -a_k$

$$\Leftrightarrow -\frac{1}{a_{k+1}} > -\frac{1}{a_k}$$

$$\Leftrightarrow 3 - \frac{1}{a_{k+1}} > 3 - \frac{1}{a_k}$$

$$\Leftrightarrow a_{k+2} > a_{k+1}$$

so $\{a_n\}$ is increasing and bounded.
Thus it is convergent.

Thus, $\lim_{n \rightarrow \infty} (3 - \frac{1}{a_n})$ exists. Call it L .

$$\lim_{n \rightarrow \infty} (3 - \frac{1}{a_n}) = L$$

It's also true that $\lim_{n \rightarrow \infty} a_{n+1} = L$.

Recall $a_{n+1} = 3 - \frac{1}{a_n}$

$$L = 3 - \frac{1}{L} \Rightarrow L^2 = 3L - 1$$
$$L^2 - 3L + 1 = 0 \Rightarrow L = \frac{3 \pm \sqrt{9-4}}{2}$$

But $\frac{3-\sqrt{5}}{2} < 1$

So $L = \frac{3+\sqrt{5}}{2}$

Mathematical induction : If $n = k+1$, this would be
 $\downarrow \frac{(k+1)(k+2)}{2}$

Consider $1 + 2 + \dots + n = \frac{n(n+1)}{2}$, $n=1, 2, 3, \dots$

Step 1: Show statement is true for some specific value of n . Usually $n=1$.

$$1 = \frac{1(1+1)}{2} \Rightarrow 1 = 1 \checkmark$$

Step 2: Assume statement is true for some number k . That is, assume

$$1 + 2 + \dots + k = \frac{k(k+1)}{2}$$

Step 3: Show that being true for k means it's also true for $k+1$.

$$1 + 2 + \dots + k + k+1 = \frac{k(k+1)}{2} + k+1$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}, \text{ as expected.}$$