11.3 - The Integral Test and Estimates of Sums

Preparation (Watch at Home)

11.3a

The Integral Test:

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent $\iff \int_1^{\infty} f(x)\,dx$ is convergent.

Convergence of a p-series

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$

11.3b

Exercise:

Determine whether the series is convergent or divergent.

$$\frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \cdots$$

11.3c

Estimating sums

The remainder of a partial sum and estimating sums: Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent with sum s. Then if S_n is a partial sum,

$R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$ is the remainder in approximating s

$\int_{n+1}^{\infty} f(x)\, dx \leq R_n \leq \int_{n}^{\infty} f(x)\, dx$

$s_n + \int_{n+1}^{\infty} f(x)\, dx \leq s \leq s_n + \int_{n}^{\infty} f(x)\, dx$

11.3e

The Integral Test: Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent $\Longleftrightarrow \int_{1}^{\infty} f(x)\, dx$ is convergent.

Proof:

In-Class Discussion

11.3d

Exercise:

a) Find the partial sum s_{10} of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$. Estimate the error in using s_{10} as an approximation to the sum of the series.

b) Use $s_n + \int_{n+1}^{\infty} f(x)\,dx \le s \le s_n + \int_{n}^{\infty} f(x)\,dx$ with $n = 10$ to give an improved estimate of the sum.

c) Compare your estimate in part (b) with the exact value $\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$. (This is known as the Riemann zeta function and is used in physics and higher-level math.)

d) Find a value of n that will ensure that the error in the approximation $s \approx s_n$ is less than 0.001.

In-Class Practice

Determine whether the series is convergent or divergent.

1. $\displaystyle\sum_{n=3}^{\infty} \frac{3n-4}{n^2-2n}$

2. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

3. $\sum_{n=1}^{\infty} \frac{n}{n^4 + 1}$

4. Explain why the Integral Test can't be used to determine whether the series is convergent.

$\sum_{n=1}^{\infty} \frac{\cos^2 n}{1 + n^2}$

5. Find the sum of the series $\sum_{n=1}^{\infty} ne^{-2n}$ correct to four decimal places.