11.10 - Taylor and Maclaurin Series

Preparation (Watch at Home)

11.10a

Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Maclaurin series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

11.10b

Power series representations of more functions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

11.10c

Taylor polynomials

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Theorem (statement):

Let $R_n(x) = f(x) - T_n(x)$. If $\lim_{n\to\infty} R_n(x) = 0$, then f is equal to the sum of its Taylor series

Taylor's Inequality

If $\left|f^{(n+1)}(x)\right| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series of f satisfies $|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a| \leq d$.

11.10d

The Binomial Series

$$
\begin{aligned}
(1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n \\
&= 1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots + \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n + \cdots
\end{aligned}
$$

Exercises:

11.10e

Use the definition of a Taylor series to find the first four nonzero terms of the series for $f(x)$ centered at the given value of a.
$f(x) = \ln x$, $a = 1$

11.10g

Use the binomial series to expand the function as a power series. State the radius of convergence.

$\sqrt[3]{8+x}$

11.10i

Use multiplication or division of power series to find the first three nonzero terms in the Maclaurin series for each function.

$y = e^x \ln(1 + x)$

$y = \sec x$

11.10j

Evaluate the indefinite integral as an infinite series.
$\int x^2 \sin\left(x^2\right)\, dx$

In-Class Discussion

11.10f

Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.] Also find the associated radius of convergence.
$f(x) = e^{-2x}$

11.10h

Use a known Maclaurin series to obtain the Maclaurin series for the given function.

$f(x) = e^{3x} - e^{2x}$

$f(x) = x^2 \ln(1 + x^3)$

Use series to evaluate the limit.

$$\lim_{x \to 0} \frac{1 - \cos x}{1 + x - e^x}$$

Find the sum of the series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$$

In-Class Practice

1. Use the definition of a Taylor series to find the first four nonzero terms of the series for $f(x)$ centered at the given value of a.

$f(x) = \cos^2 x,\ a = 0$

2. Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. [Assume that f has a power series expansion. Do not show that $R_n \to 0$.] Also find the associated radius of convergence.

$f(x) = x \cos x$

Find the Taylor series for $f(x)$ at the given value of a. [Assume that f has a power series expansion. Do not show that $R_n \to 0$.] Also find the associated radius of convergence.

3. $f(x) = \sqrt{x}$, $a = 16$

4. Evaluate the indefinite integral as an infinite series.

$\int \arctan(x^2)\,dx$

5. Use series to approximate the definite integral $\int_0^{1/2} x^3 \arctan x \, dx$ accurate to four decimal places.

6. Use multiplication or division of power series to find the first three nonzero terms in the Maclaurin series for each function.

$y = e^x \sin^2 x$