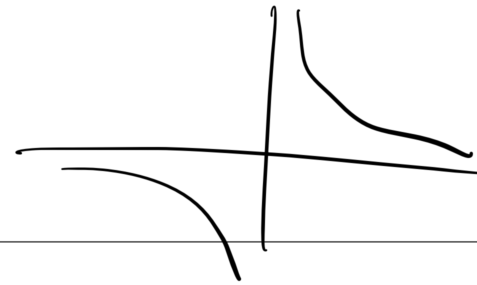
6.4 - Derivatives of Logarithmic Functions

Preparation (Watch at Home)

6.4a

Calculus of the natural logarithmic function

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$



Chain Rule

$$\frac{d}{dx}\left[\ln(u(x))\right] = \frac{u'(x)}{u(x)}$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\frac{d}{dx}\left[\ln|x|\right] = \frac{1}{x}, \; x \neq 0$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Exercise:

Show that $\int \tan x \, dx = \ln|\sec x| + C$.

6.4b

Exercises:

Differentiate the function

$f(x) = \ln(\sin^2 x)$

The hard way

$$f'(x) = \frac{2\cancel{\sin x}\cos x}{\sin^2 x}$$

$$f'(x) = 2\frac{\cos x}{\sin x}$$

$$f'(x) = 2\cot x$$

The easier way

$$f(x) = \ln(\sin^2 x)$$

$$f(x) = 2\ln(\sin x)$$

$$f'(x) = 2\frac{\cos x}{\sin x}$$

$$f'(x) = 2\cot x$$

$$\sqrt{x^2-1} = (x^2-1)^{1/2} \rightarrow \frac{1}{2}(x^2-1)^{-1/2} \, 2x$$

$h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$

$$h'(x) = \frac{1 + \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2-1}}}{x+\sqrt{x^2-1}} \quad \frac{\sqrt{x^2-1}}{\sqrt{x^2-1}}$$

$$h'(x) = \frac{\overset{1}{\cancel{\sqrt{x^2-1} + x}}}{\cancel{\left(x+\sqrt{x^2-1}\right)}\sqrt{x^2-1}} = \frac{1}{\sqrt{x^2-1}}$$

$y = \ln(\csc x - \cot x)$

6.4c

Exercises:

Evaluate the integral.

$$\int_0^3 \frac{dx}{5x+1}$$

$$\int \frac{\cos x}{2 + \sin x}\, dx$$

6.4d

Calculus of logs and exponentials with bases other than e

$$\frac{d}{dx}\left(\log_b x\right) = \frac{1}{x \ln b}$$

$$\frac{d}{dx}(b^x) = \ln b \cdot b^x$$

$$\int b^x \, dx = \frac{b^x}{\ln b} + C$$

6.4f

Exercise:

Use logarithmic differentiation to find the derivative of the function.

$$y = \frac{e^{-x} \cos^2 x}{x^2 + x + 1}$$

In-Class Discussion

6.4e

Definition: (Euler's number e)

Recall that $\frac{d}{dx}(b^x) = b^x \boxed{\lim_{h\to 0}\frac{b^h - b^0}{h-0}}$

$\Rightarrow \lim_{h\to 0}\frac{b^h-1}{h} = \ln b$

Recall that if $b=e$, then $\lim_{x\to 0}\frac{b^x-1}{x} = 1$

For $f(x)=\ln x$, $f'(x)=\frac{1}{x} \Rightarrow \boxed{f'(1)=1}$

But $f'(1) = \lim_{\cancel{h}\,x\to 0}\frac{f(1+\cancel{h}\,x)-f(1)}{\cancel{h}\,x}$

$f'(1) = \lim_{x\to 0}\frac{\ln(1+x) - \cancel{\ln 1}\,0}{x} = \lim_{x\to 0}\frac{\ln(1+x)}{x}$

$$\boxed{f'(1) = \lim_{x\to 0}\frac{1}{x}\ln(1+x)}$$

$$e = \lim_{x\to 0}(1+x)^{\frac{1}{x}}$$

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$1 = \lim_{x \to 0} \ln (1+x)^{1/x}$$

$$e^{1} = \cancel{e^{\ln}} \lim_{x \to 0} (1+x)^{1/x}$$

$$e = \lim_{x \to 0} (1+x)^{1/x}$$

Suppose we let $n = \frac{1}{x}$.

Then $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n}$

6.4g

Exercise:

Use logarithmic differentiation to find the derivative of the function.

$y = x^{\cos x}$

Process for logarithmic differentiation

Take the natural log of both sides

Use properties of logs

$\frac{y'}{y}$ or $\frac{1}{y}y'$

Differentiate (usually with respect to x)—use implicit differentiation for y

Give $\frac{dy}{dx}$ or y' in terms of x (or whatever the independent variable is)

Summary of techniques involving exponents

(variable base)$^{\text{constant exponent}}$ –power rule (ex: $y = x^2$)

(constant base)$^{\text{variable exponent}}$ –rules for exponentials (ex: $y = e^x$ or $y = 2^x$)

(variable base)$^{\text{variable exponent}}$ –logarithmic differentiation

In-Class Practice

Differentiate the function.

1. $g(t) = \sqrt{1 + \ln t}$ $= (1+\ln t)^{1/2}$

$g'(t) = \dfrac{1/t}{2\sqrt{1+\ln t}} \dfrac{t}{t}$

$g'(t) = \dfrac{1}{2t\sqrt{1+\ln t}}$

2. $g(x) = x \sin(2^x)$ $\qquad \frac{d}{dx}(b^x) = \ln b \; b^x$

$u = x \Rightarrow u' = 1 \qquad v = \sin(2^x) \Rightarrow v' = \cos(2^x) \cdot 2^x \ln 2$

$g' = \sin(2^x) + x \cos(2^x)\, 2^x \ln 2$

3. $F(t) = 3^{\cos 2t}$

Use logarithmic differentiation to find the derivative of the function.

4. $y = \sqrt{x}e^{x^2-x}(x+1)^{\frac{2}{3}}$

$$\ln y = \ln\left[\sqrt{x}\, e^{x^2-x} (x+1)^{2/3}\right]$$

$$\ln y = \ln \underbrace{\sqrt{x}}_{x^{1/2}} + \underline{\ln e^{x^2-x}} + \ln (x+1)^{2/3}$$

$$\ln y = \frac{1}{2}\ln x + x^2 - x + \frac{2}{3}\ln(x+1)$$

$$\frac{1}{y}y' = \frac{1}{2x} + 2x - 1 + \frac{2}{3(x+1)}$$

$$y' = y\left[\frac{1}{2x} + 2x - 1 + \frac{2}{3(x+1)}\right]$$

$$y' = \sqrt{x}\, e^{x^2-x} (x+1)^{2/3}\left[\frac{1}{2x} + 2x - 1 + \frac{2}{3(x+1)}\right]$$

5. $y = (\sin x)^{\ln x}$

$$\ln y = \ln (\sin x)^{\ln x}$$

$$\ln y = \underbrace{\ln x}\,\underbrace{\ln(\sin x)}$$

$$\frac{1}{y}y' = \frac{1}{x}\ln(\sin x) + \ln x \,\frac{\cos x}{\sin x}$$

$$y' = y\left[\frac{1}{x}\ln(\sin x) + \ln x \cot x\right]$$

$$y' = (\sin x)^{\ln x}\left[\frac{1}{x}\ln(\sin x) + \ln x \cot x\right]$$

Evaluate the integral.

6. $\int \frac{\cos(\ln t)}{t}\, dt$

$$u = \ln t \Rightarrow du = \frac{1}{t}dt$$

$$= \int \cos u \, du = \sin u + C$$

$$= \sin(\ln t) + C$$

7. $\displaystyle\int \frac{e^x}{e^x+1}\,dx$ $u = e^x + 1 \Rightarrow du = e^x dx$

$$\int \frac{1}{u}\,du = \ln|u| + C$$

$$= \ln(e^x+1) + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int_{-5}^{-3} \frac{1}{x}\,dx = \ln|x| \Big|_{-5}^{-3}$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\csc x] = -\csc x \cot x$$