6.2 - Exponential Functions and Their Derivatives

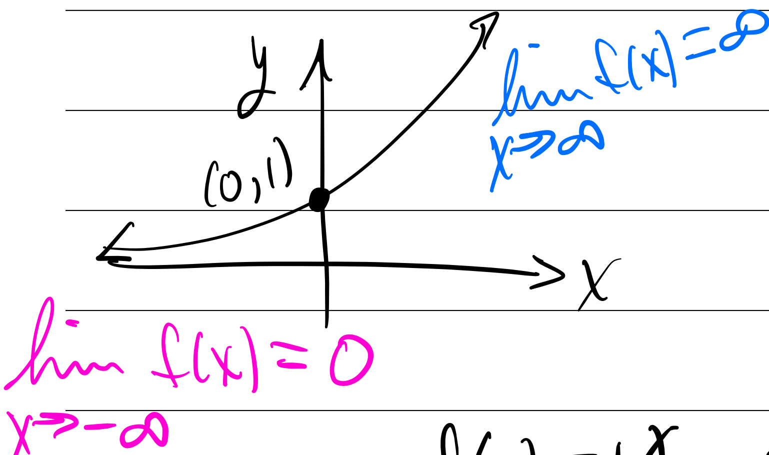
Preparation (Watch at Home)

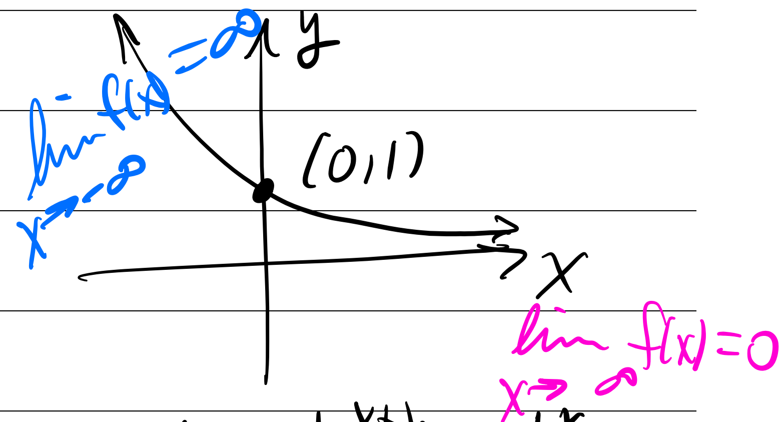
6.2a

The derivative of an exponential function using the definition of a derivative

$f(x)=b^x$, $b>0$, $b\neq 1$, $x\in\mathbb{R}$

$b>1$ $\qquad\qquad$ $0<b<1$





For $f(x)=b^x$, $f'(x)=\lim_{h\to0}\frac{b^{x+h}-b^x}{h}$

$$f'(x)=\lim_{h\to0}\frac{b^x(b^h-1)}{h}$$

We'll revisit this limit $\quad f'(x)=b^x\lim_{h\to0}\frac{b^h-1}{h}$ $\quad$ constant multiple of the function

The derivative of the natural exponential function

Note that $b^0=1$, so $\lim_{h\to0}\frac{b^h-b^0}{h-0}=f'(0)$

So $f'(x) = f(x)\, f'(0)$

Consider $b=2$ & $b=3$

$\lim_{h\to 0} \frac{2^h - 1}{h}$

h	$\frac{2^h-1}{h}$
0.001	0.693387
10^{-7}	0.69315

less than 1

$\lim_{h\to 0} \frac{3^h - 1}{h} \approx 1.1$ larger than 1

For $b = e \approx 2.718$, $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$

So if $f(x) = e^x$, then $f'(x) = e^x \cdot 1$

That is, $\frac{d}{dx}(e^x) = e^x$.

The Chain Rule applies: $\frac{d}{dx}\left[e^{u(x)}\right] = u'(x)e^{u(x)}$

$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}$ $\qquad$ $f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h-0}$

Exercise:

Find y' and y'' for $y = e^{x2}$.

e^{x^2}

$y' = 2xe^{x^2}$

For y'', $\quad u = 2x \quad v = e^{x^2}$

$u' = 2 \quad v' = 2xe^{x^2}$

$y'' = 2e^{x^2} + 4x^2e^{x^2}$

A derivation of the value of Euler's number, *e*

The integral of the natural exponential function

Since $\frac{d}{dx}(e^x) = e^x$, $\int e^x dx = e^x + C$

6.2b

Exercises:

Differentiate the function.

$$y = \frac{e^x}{1 - e^x}$$

Evaluate the integral.

$$\int x^2 e^{x^3}\, dx$$

Find the limit.

$$\lim_{x \to -\infty} (1.001)^x$$

$$\lim_{x\to(\pi/2)^+} e^{\tan x}$$

In-Class Discussion

6.2c

Exercise:

Find the absolute maximum and absolute minimum values of f on the given interval.

$f(x) = xe^{\frac{x}{2}}$, $[-3,\ 1]$

$u = x \quad v = e^{x/2}$

$u' = 1 \quad v' = \frac{1}{2}e^{x/2}$

Endpoints: $f(-3) = -3e^{-3/2} \approx -0.67$

$f(1) = e^{1/2} \approx 1.65$ abs max

$f'(x) = e^{x/2} + \frac{x}{2}e^{x/2}$

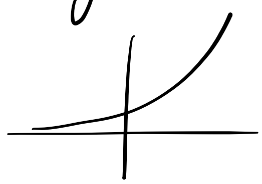
critical #: $f'(x)=0 \Rightarrow e^{x/2}+\frac{x}{2}e^{x/2}=0$

$$e^{x/2}\left(1+\frac{x}{2}\right)=0$$

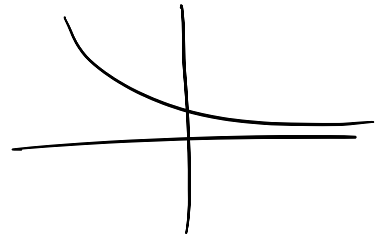
$x=-2$ is a critical #.

$f(-2)=-2e^{-1}\approx -0.74$ abs min

$y=e^{x^2}$



$y=e^{-x^2}=\frac{1}{e^{x^2}}$



In-Class Practice

Find the limit.

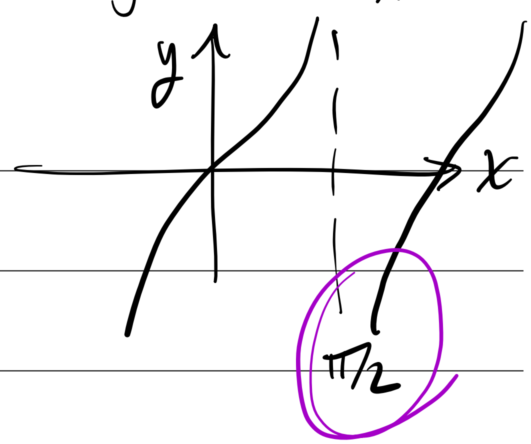
1. $\lim\limits_{x\to\infty} e^{-x^2}$

0

2. $\lim\limits_{x\to\left(\frac{\pi}{2}\right)^+} e^{\tan x} = e^{\lim_{x\to(\pi/2)^+}\tan x} \to \frac{1}{e^{\infty}}$

$= 0$

$y=\tan x$



Using that if f is cont.,

$\lim\limits_{x\to a} f(g(x)) = f\left(\lim\limits_{x\to a} g(x)\right)$

Differentiate the function.

3. $g(x)=e^{x^2-x}$

$g'(x)=(2x-1)e^{x^2-x}$

4. $V(t) = \dfrac{4+t}{te^t}$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$u = 4+t \qquad v = te^t$

$u' = 1 \qquad v' = e^t + te^t$

$$V'(t) = \frac{t\cancel{e^t} - \cancel{e^t}(4+t)(1+t)}{t^2 e^{\cancel{2}t}} = \frac{-t^2 - 4t - 4}{t^2 e^t}$$

$t^2 + 5t + 4$

5. $y = e^{\sin 2x} + \sin\left(e^{2x}\right)$

$$y' = 2\cos 2x\, e^{\sin 2x} + 2e^{2x}\cos(e^{2x})$$

Evaluate the integral.

6. $\int \boxed{e^{\sin\theta}} \cos\theta \, d\theta$

$$= \int e^u \, du \qquad u = \sin\theta \Rightarrow du = \cos\theta d\theta$$

$$= e^{\sin\theta} + C$$

7. $\int_0^1 \dfrac{\sqrt{1+e^{-x}}}{e^x}\, dx$

$$u = 1 + \frac{1}{e^x}$$

$$-\int_0^1 - \frac{\sqrt{1+e^{-x}}}{e^x} dx \qquad du = -e^{-x} dx$$

$$= -\int_0^1 \sqrt{1+e^{-x}}\, \frac{1}{e^x} dx \qquad -\frac{1}{e^x} dx$$

$$= -\int_{2}^{1+\frac{1}{e}} \sqrt{u}\,du = -\int_{2}^{1+\frac{1}{e}} u^{1/2}\,du$$

Limits

x	$u = 1 + e^{-x}$
1	$1+\frac{1}{e}$
0	2

$$= -\frac{2}{3}\,u^{3/2}\Big|_{2}^{1+\frac{1}{e}}$$

$$= -\frac{2}{3}\left(1+\frac{1}{e}\right)^{3/2} + \frac{2}{3}(2)^{3/2}$$