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# 6.1 - Inverse Functions

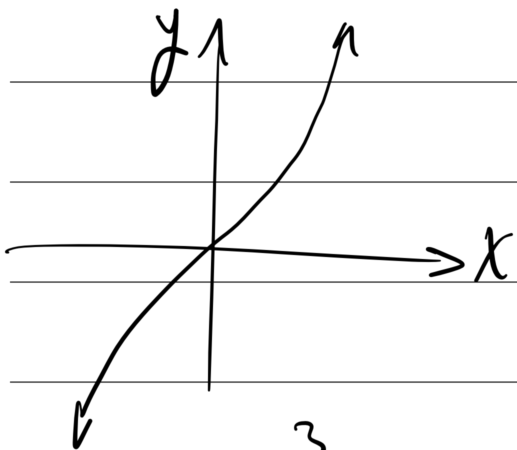
## Preparation (Watch at Home)

## 6.1a

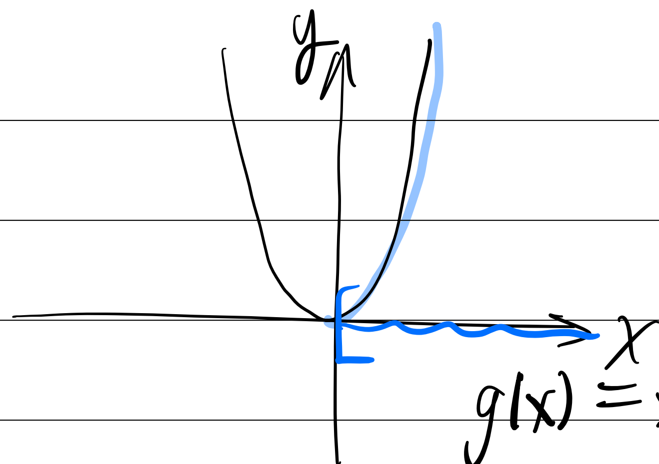
**Definition:** (one-to-one)

A function $f$ is one-to-one if $f(x_2) \neq f(x_1)$ whenever $x_2 \neq x_1$.

Description of inverse functions




$f(x) = x^3$




$g(x) = x^2$

By restricting the domain $\{x \mid x \geq 0\}$, we have a piece that is one-to-one

| $x$ | $f(x)$ |
|---|---|
| 1 | 1 |
| 2 | 8 |
| 3 | 27 |

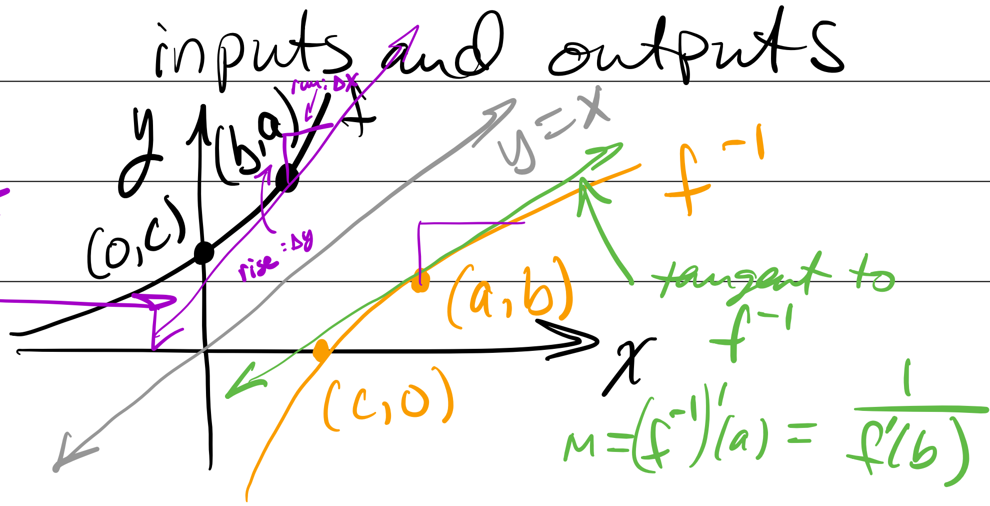
The inverse of a function reverses inputs and outputs

inputs

outputs

tangent line to $f$

slope: $m = f'(b)$




tangent to $f^{-1}$

$m = (f^{-1})'(a) = \dfrac{1}{f'(b)}$

$f(b)=a \Rightarrow b=f^{-1}(a)$

# 6.1b

so $(f^{-1})'(a)=\dfrac{1}{f'(f^{-1}(a))}$

**Theorems (statement and proof):**

If $f$ is one-to-one and continuous on $[a,\ b]$, then $f^{-1}$ is also continuous.

$f^{-1}(f(x)) = f^{-1}(y) \Rightarrow x = f^{-1}(y)$

$f(y) = \cancel{f}(\cancel{f^{-1}}(x))$

$$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}$$

If $y = f^{-1}(x)$, then

$f(y) = x$

Implicit differentiation: $f'(y)\dfrac{dy}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{f'(y)} \Rightarrow \frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))}$$

## 6.1c

**Exercises:**

If $f(x) = x^5 + x^3 + x$, find $f^{-1}(3)$ and $f\left(f^{-1}(2)\right)$.

Find $\left(f^{-1}\right)'(a)$ for $f(x) = x^3 + 3\sin x + 2\cos x$ and $a = 2$.

$$2 = x^3 + 3\sin x + 2\cos x$$

Conditional: if p, then q. $p \Rightarrow q$

Contrapositive: if not q, then not p.

## In-Class Discussion

### 6.1d

For a one-to-one function f, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

**Exercise:**

$= (x-2)^{1/2}$

Let $f(x) = \sqrt{x-2}$ and $a = 2$.

a) Show that $f$ is one-to-one

Assume $f(b) = f(a)$. Then $\sqrt{b-2} = \sqrt{a-2}$

$\Rightarrow b-2=a-2 \Rightarrow b=a$

OR. $f'(x)=\frac{1}{2}(x-2)^{-1/2}=\frac{1}{2\sqrt{x-2}}>0$

$\Rightarrow f$ is one-to-one

Fail: $g(x)=x^2$ $g(b)=g(a) \Rightarrow b^2=a^2 \Rightarrow b=\pm a$

**Find** $\left(f^{-1}\right)'(a)=\dfrac{1}{f'\left(f^{-1}(a)\right)}$

Need • $f'(x)=\frac{1}{2\sqrt{x-2}}$

• $f^{-1}(a)$

$y=f(x)$

$f^{-1}(y)=x$

$f(x)=\sqrt{x-2} \Rightarrow f^{-1}(2)$

the x-value that gives $y=2$

$2=\sqrt{x-2} \Rightarrow 4=x-2 \Rightarrow x=6$

so $f^{-1}(2)=6$

$(f^{-1})'(2)=\frac{1}{f'(f^{-1}(2))}=\frac{1}{f'(6)}$ $f'(6)=\frac{1}{4}$

$(f^{-1})'(2)=4$

**In-Class Practice**

1. Let $f(x)=\dfrac{1}{x-1}$, $x>1$, $a=2$.

a) Show that $f$ is one-to-one.

b) Use Theorem 7 to find $(f^{-1})'(a)$.

c) Calculate $f^{-1}(x)$ and state the domain and range of $f^{-1}$.

d) Calculate $(f^{-1})'(a)$ from the formula in part (c) and check that it agrees with the result of part (b).

e) Sketch the graphs of $f$ and $f^{-1}$ on the same axes.

2. Find $(f^{-1})'(a)$ for $f(x) = \sqrt{x^3 + 4x + 4}, \quad a = 3.$

$\cdot f^{-1}(3)$

$3 = \sqrt{x^3+4x+4} \Rightarrow 9 = x^3+4x+4$

If $x=1$, $x^3+4x+4=9$

$\cdot f'(x)$ $f'(x) = \frac{3x^2+4}{2\sqrt{x^3+4x+4}}$

$f'(1) = \frac{7}{2\sqrt{9}} = \frac{7}{6}$

Formula $(f^{-1})'(3) = \frac{1}{f'(1)} = \frac{6}{7}$

3. If $g$ is an increasing function such that $g(2) = 8$ and $g'(2) = 5$, calculate $(g^{-1})'(8)$.

$\rightarrow g^{-1}(8) = 2$

$(g^{-1})'(8) = \frac{1}{g'(g^{-1}(8))} = \frac{1}{g'(2)} = \frac{1}{5}$

4. If $f$ is a one-to-one, twice differentiable function with inverse function $g$, show that

$$g''(x) = -\frac{f''(g(x))}{[f'(g(x))]^3}$$

What conclusion can you draw about the graph of $f^{-1}(x)$ if $f$ is increasing and concave upward?